# IS-ZC444: ARTIFICIAL INTELLIGENCE

## Lecture-14: Bayesian Network + Regression


**Dr. Kamlesh Tiwari**
Assistant Professor
Department of Computer Science and Information Systems,
BITS Pilani, Pilani, Jhunjhunu-333031, Rajasthan, INDIA

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- The process of computing a marginal from a joint distribution is called marginalisation.

$$p(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) = \sum_{x_i} p(x_1, x_2, ..., x_n)$$

# Let's see this

$$\sum_{J} \left( p(J|R) \times f(R) \right) =$$

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$$\sum_J \left(p(J|R) \times f(R)\right) = f(R)$$

**Proof:**

$$\begin{aligned}
\sum_J \left(p(J|R) \times f(R)\right) &= \sum_J \left(\frac{p(J,R)}{p(R)} \times f(R)\right) \\
&= \frac{\sum_J \left(p(J,R) \times f(R)\right)}{p(R)} \\
&= \frac{f(R) \times \sum_J p(J,R)}{p(R)} \\
&= \frac{f(R) \times p(R)}{p(R)} \\
&= f(R)
\end{aligned}$$

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- Belief networks are also called as *Bayes' Networks* or *Bayesian Belief Networks*

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- Model of Traceya's world involves probability distribution on $T, J, R, S$ that has $2^4 = 16$ states

# Modeling Independencies (contd..)

- However we know

$$p(T, J, R, S) = p(T|J, R, S)p(J, R, S)$$

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- With $p(T=1|J,R,S)$, one can use normalization to compute $p(T=0|J,R,S)$ as $1 - p(T=1|J,R,S)$
- Computation of other factors would also need 4+2+1 values
- Total we need 8+4+2+1=15 values

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- We may assume that Traceya's grass is wet depends only directly on whether or not it has been raining and whether or not her sprinkler was on so $p(T|J,R,S) = p(T|R,S)$

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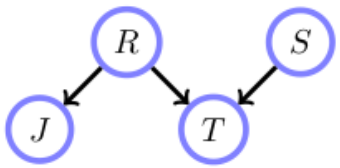
- Number of values we need to specify is 4+2+1+1=8
- We can represent these conditional independencies as



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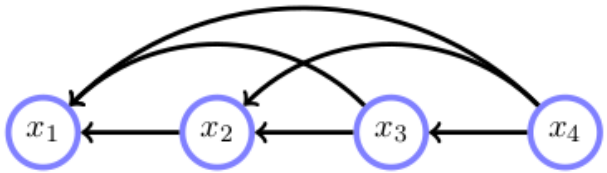
- Represented as a directed graph, with an arrow pointing from a parent variable to child variable, a belief network corresponds to a Directed Acyclic Graph (DAG)



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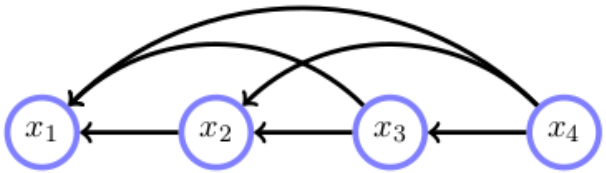


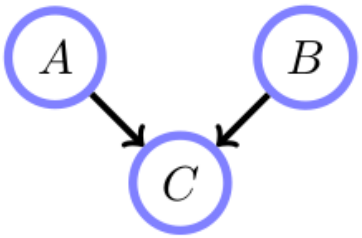


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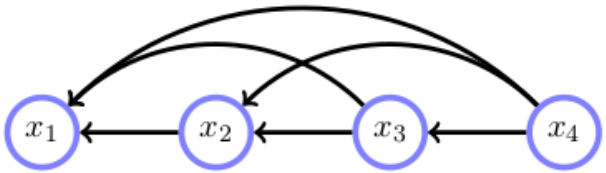


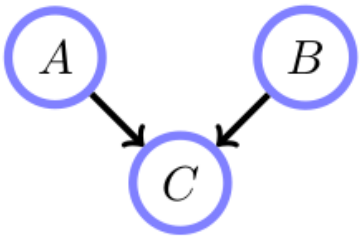


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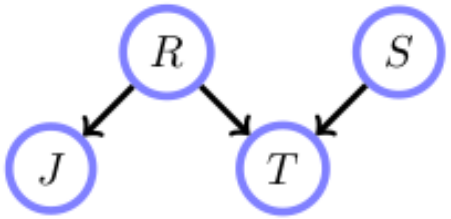

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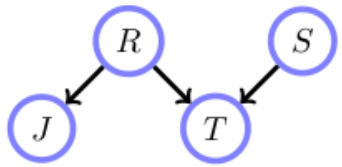
Using following Beleif Network; calculate




1. Probability that the sprinkler was *on* overnight, given that Traceya's grass is wet.

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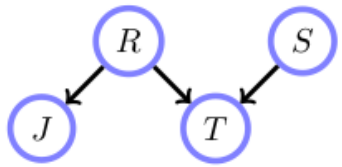


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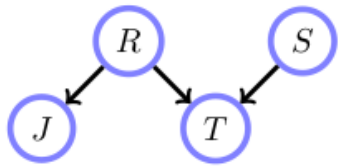


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2. Probability that Traceya's sprinkler was *on* overnight, given that her grass is wet and that Jack's grass is also wet.

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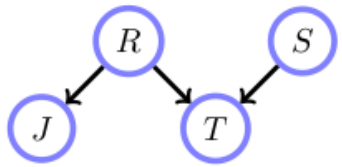


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2. Probability that Traceya's sprinkler was *on* overnight, given that her grass is wet and that Jack's grass is also wet. $p(S{=}1|T{=}1, J{=}1) = 0.1604$

# Example: Probability of $p(S{=}1|T{=}1)$

$$
\begin{aligned}
p(S=1|T=1) &= \frac{p(S=1,T=1)}{p(T=1)} \\
&= \frac{\sum_{J,R} p(S=1,J,R,T=1)}{\sum_{J,R,S} p(T=1,J,R,S)} \qquad (1) \\
&= \frac{\sum_{J,R} p(J|R)p(T=1|R,S=1)p(R)p(S=1)}{\sum_{J,R,S} p(J|R)p(T=1|R,S)p(R)p(S)} \\
&= \frac{\sum_{J,R} p(T=1|R,S=1)p(R)p(S=1)}{\sum_{J,R,S} p(T=1|R,S)p(R)p(S)} \qquad (2) \\
&= \frac{0.9\times 0.8\times 0.1 + 1\times 0.2\times 0.1}{.9\times .8\times .1 + 1\times .2\times .1 + 0\times .8\times .9 + 1\times .2\times .9} \\
&= \boxed{0.3382}
\end{aligned}
$$

Uses given belief network in (1) and proof in (2)

# Regression

**Regression** predicts value of continuous a target variable

| $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|
| 10 | 50 | 20 | 10 |
| 11 | 31 | 22 | 12 |
| 11 | 12 | 15 | 4 |
| 20 | 55 | 20 | 22 |
| 23 | 41 | 27 | 1 |
| 31 | 12 | 35 | 9 |
| 13 | 18 | 12 | 23 |
| 21 | 55 | 16 | 16 |
| 32 | 56 | 27 | 22 |
| 8 | 22 | 35 | ?? |

What should come at the place of ??

# Regression

**Regression** predicts value of continuous a target variable

# Regression

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- A simplest model for regression can be a linear combination of the input variables

$$y(x, w) = w_0 + w_1 x_1 + ... + w_n x_n$$

where $x$ is a $n$ dimensional vector $(x_1, x_2, ..., x_n)$ representing some feature

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- It can be extended by considering linear combination of fixed nonlinear functions $\phi$ (called basis functions)

$$y(x, w) = w_0 + \sum_{i=1}^{n} w_i \phi_i(x)$$

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- It can be extended by considering linear combination of fixed nonlinear functions $\phi$ (called basis functions)

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- In short $y(x, w) = w^T \phi(x)$

# Regression

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- A simplest model for regression can be a linear combination of the input variables

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- It can be extended by considering linear combination of fixed nonlinear functions $\phi$ (called basis functions)

$$y(x, w) = w_0 + \sum_{i=1}^{n} w_i \phi_i(x)$$

- In short $y(x, w) = w^T \phi(x)$
- Objective is to choose $w$ such that it makes $y(x^{(i)}, w)$ as close to $y^{(i)}$ as possible

# Our Regression Example

- If we could correct estimate the values of $w$'s we could determine $y(x^{(i)}, w)$ for all values

| $x_1$ | $x_2$ | $x_3$ | $y$ | $y(x^{(i)}, w)$ |
|---|---|---|---|---|
| 10 | 50 | 20 | 10 | 8 |
| 11 | 31 | 22 | 12 | 9 |
| 11 | 12 | 15 | 4 | 3 |
| 20 | 55 | 20 | 22 | 26 |
| 23 | 41 | 27 | 1 | 1 |
| 31 | 12 | 35 | 9 | 4 |
| 13 | 18 | 12 | 23 | 30 |
| 21 | 55 | 16 | 16 | 13 |
| 32 | 56 | 27 | 22 | 21 |
| 8 | 22 | 35 | ?? | 6 |

Now the question is that how good this $w$ is?

# Regression

- Determining $w$, is similar to solving a minimization problem. Let us define a **squared error cost function** as

$$J(w) = \frac{1}{2m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)})^2$$

where $m$ is number of training examples

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- Then one have to minimize the value of $J(w)$

$$\underset{w}{\operatorname{argmin}}\, J(w)$$

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$$\underset{w}{\operatorname{argmin}}\, J(w)$$

- Basic idea: Push $w_i$ a bit against the direction of its gradient

# Linear Regression

| $x_1$ | $x_2$ | $x_3$ | $y$ | $y(x^{(i)}, w)$ | $(y(x^{(i)}, w) - y)^2$ |
|---|---|---|---|---|---|
| 10 | 50 | 20 | 10 | 8 | 4 |
| 11 | 31 | 22 | 12 | 9 | 9 |
| 11 | 12 | 15 | 4 | 3 | 1 |
| 20 | 55 | 20 | 22 | 26 | 16 |
| 23 | 41 | 27 | 1 | 1 | 0 |
| 31 | 12 | 35 | 9 | 4 | 25 |
| 13 | 18 | 12 | 23 | 30 | 49 |
| 21 | 55 | 16 | 16 | 13 | 9 |
| 32 | 56 | 27 | 22 | 21 | 1 |

Assume for some $w$ we computed $y(x^{(i)}, w)$ then

$$\begin{aligned} J(w) &= \frac{1}{2 \times 9} \times 114 \\ &= 6.33 \end{aligned}$$

# Gradient Descent

**Algorithm 1:** Gradient Descent

1 Initialize $w$ randomly
2 **repeat**
3 | Simultaneously update all $w_j$ with $w_j - \alpha \frac{\partial}{\partial w_j} J(w)$
4 **until** *converge*;
5 **return** $w$

# Gradient Descent

**Algorithm 2:** Gradient Descent

1 Initialize $w$ randomly
2 **repeat**
3 | Simultaneously update all $w_j$ with $w_j - \alpha \frac{\partial}{\partial w_j} J(w)$
4 **until** *converge*;
5 **return** $w$

- Here $\alpha$ is a learning rate. If $\alpha$ is small enough then $J(w)$ would decrease in every iteration

# Gradient Descent

**Algorithm 3:** Gradient Descent

1 Initialize $w$ randomly
2 **repeat**
3 | Simultaneously update all $w_j$ with $w_j - \alpha \frac{\partial}{\partial w_j} J(w)$
4 **until** *converge*;
5 **return** $w$

- Here $\alpha$ is a learning rate. If $\alpha$ is small enough then $J(w)$ would decrease in every iteration
(large $\alpha$ can overshoot the minimum and may fail to converge)

# Gradient Descent

**Algorithm 4:** Gradient Descent

1 Initialize $w$ randomly
2 **repeat**
3 | Simultaneously update all $w_j$ with $w_j - \alpha \frac{\partial}{\partial w_j} J(w)$
4 **until** *converge*;
5 **return** $w$

- Here $\alpha$ is a learning rate. If $\alpha$ is small enough then $J(w)$ would decrease in every iteration
  (large $\alpha$ can overshoot the minimum and may fail to converge)
- Susceptible to local minimum

# Gradient Descent

**Algorithm 5:** Gradient Descent

1 Initialize $w$ randomly
2 **repeat**
3 | Simultaneously update all $w_j$ with $w_j - \alpha \frac{\partial}{\partial w_j} J(w)$
4 **until** *converge*;
5 **return** $w$

- Here $\alpha$ is a learning rate. If $\alpha$ is small enough then $J(w)$ would decrease in every iteration
  (large $\alpha$ can overshoot the minimum and may fail to converge)
- Susceptible to local minimum
- As it moves closer to local minimum, it automatically takes smaller steps as gradient decreases

# Partial Derivative term

$$\frac{\partial}{\partial w_j} J(w) = \frac{\partial}{\partial w_j} \frac{1}{2m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)})^2$$

# Partial Derivative term

$$\begin{aligned}\frac{\partial}{\partial w_j} J(w) &= \frac{\partial}{\partial w_j} \frac{1}{2m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)})^2 \\ &= \frac{1}{2m} \sum_{i=1}^{m} \frac{\partial}{\partial w_j} (y(x^{(i)}, w) - y^{(i)})^2\end{aligned}$$

# Partial Derivative term

$$
\begin{aligned}
\frac{\partial}{\partial w_j} J(w) &= \frac{\partial}{\partial w_j} \frac{1}{2m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)})^2 \\
&= \frac{1}{2m} \sum_{i=1}^{m} \frac{\partial}{\partial w_j} (y(x^{(i)}, w) - y^{(i)})^2 \\
&= \frac{1}{2m} \sum_{i=1}^{m} 2(y(x^{(i)}, w) - y^{(i)}) \frac{\partial}{\partial w_j} (y(x^{(i)}, w) - y^{(i)})
\end{aligned}
$$

# Partial Derivative term

$$
\begin{aligned}
\frac{\partial}{\partial w_j} J(w) &= \frac{\partial}{\partial w_j} \frac{1}{2m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)})^2 \\
&= \frac{1}{2m} \sum_{i=1}^{m} \frac{\partial}{\partial w_j} (y(x^{(i)}, w) - y^{(i)})^2 \\
&= \frac{1}{2m} \sum_{i=1}^{m} 2(y(x^{(i)}, w) - y^{(i)}) \frac{\partial}{\partial w_j} (y(x^{(i)}, w) - y^{(i)}) \\
&= \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) \frac{\partial}{\partial w_j} y(x^{(i)}, w)
\end{aligned}
$$

# Partial Derivative term

$$
\begin{aligned}
\frac{\partial}{\partial w_j} J(w) &= \frac{\partial}{\partial w_j} \frac{1}{2m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)})^2 \\
&= \frac{1}{2m} \sum_{i=1}^{m} \frac{\partial}{\partial w_j} (y(x^{(i)}, w) - y^{(i)})^2 \\
&= \frac{1}{2m} \sum_{i=1}^{m} 2(y(x^{(i)}, w) - y^{(i)}) \frac{\partial}{\partial w_j} (y(x^{(i)}, w) - y^{(i)}) \\
&= \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) \frac{\partial}{\partial w_j} y(x^{(i)}, w)
\end{aligned}
$$

For $y(x^{(i)}, w) = w_0 + w_1 x_1^{(i)} + ... + w_n x_n^{(i)}$ we have

# Partial Derivative term

$$
\begin{aligned}
\frac{\partial}{\partial w_j} J(w) &= \frac{\partial}{\partial w_j} \frac{1}{2m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)})^2 \\
&= \frac{1}{2m} \sum_{i=1}^{m} \frac{\partial}{\partial w_j} (y(x^{(i)}, w) - y^{(i)})^2 \\
&= \frac{1}{2m} \sum_{i=1}^{m} 2(y(x^{(i)}, w) - y^{(i)}) \frac{\partial}{\partial w_j} (y(x^{(i)}, w) - y^{(i)}) \\
&= \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) \frac{\partial}{\partial w_j} y(x^{(i)}, w)
\end{aligned}
$$

For $y(x^{(i)}, w) = w_0 + w_1 x_1^{(i)} + ... + w_n x_n^{(i)}$ we have $\boxed{\frac{\partial}{\partial w_j} y(x^{(i)}, w) = x_j^{(i)}}$

# Partial Derivative term

$$\begin{aligned}
\frac{\partial}{\partial w_j} J(w) &= \frac{\partial}{\partial w_j} \frac{1}{2m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)})^2 \\
&= \frac{1}{2m} \sum_{i=1}^{m} \frac{\partial}{\partial w_j} (y(x^{(i)}, w) - y^{(i)})^2 \\
&= \frac{1}{2m} \sum_{i=1}^{m} 2(y(x^{(i)}, w) - y^{(i)}) \frac{\partial}{\partial w_j} (y(x^{(i)}, w) - y^{(i)}) \\
&= \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) \frac{\partial}{\partial w_j} y(x^{(i)}, w)
\end{aligned}$$

For $y(x^{(i)}, w) = w_0 + w_1 x_1^{(i)} + ... + w_n x_n^{(i)}$ we have $\boxed{\frac{\partial}{\partial w_j} y(x^{(i)}, w) = x_j^{(i)}}$

$$\frac{\partial}{\partial w_j} J(w) = \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) x_j^{(i)}$$

# Batch-Gradient Descent

**Algorithm 6:** Batch-Gradient Descent

1 Initialize $w$ randomly
2 **repeat**
3 | Simultaneously update all $w_j$ with $w_j - \alpha \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) x_j^{(i)}$
4 **until** *converge*;
5 **return** $w$

# Batch-Gradient Descent

**Algorithm 7:** Batch-Gradient Descent

1 Initialize $w$ randomly
2 **repeat**
3 | Simultaneously update all $w_j$ with $w_j - \alpha \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) x_j^{(i)}$
4 **until** *converge*;
5 **return** $w$

- At every step it evaluate all training examples

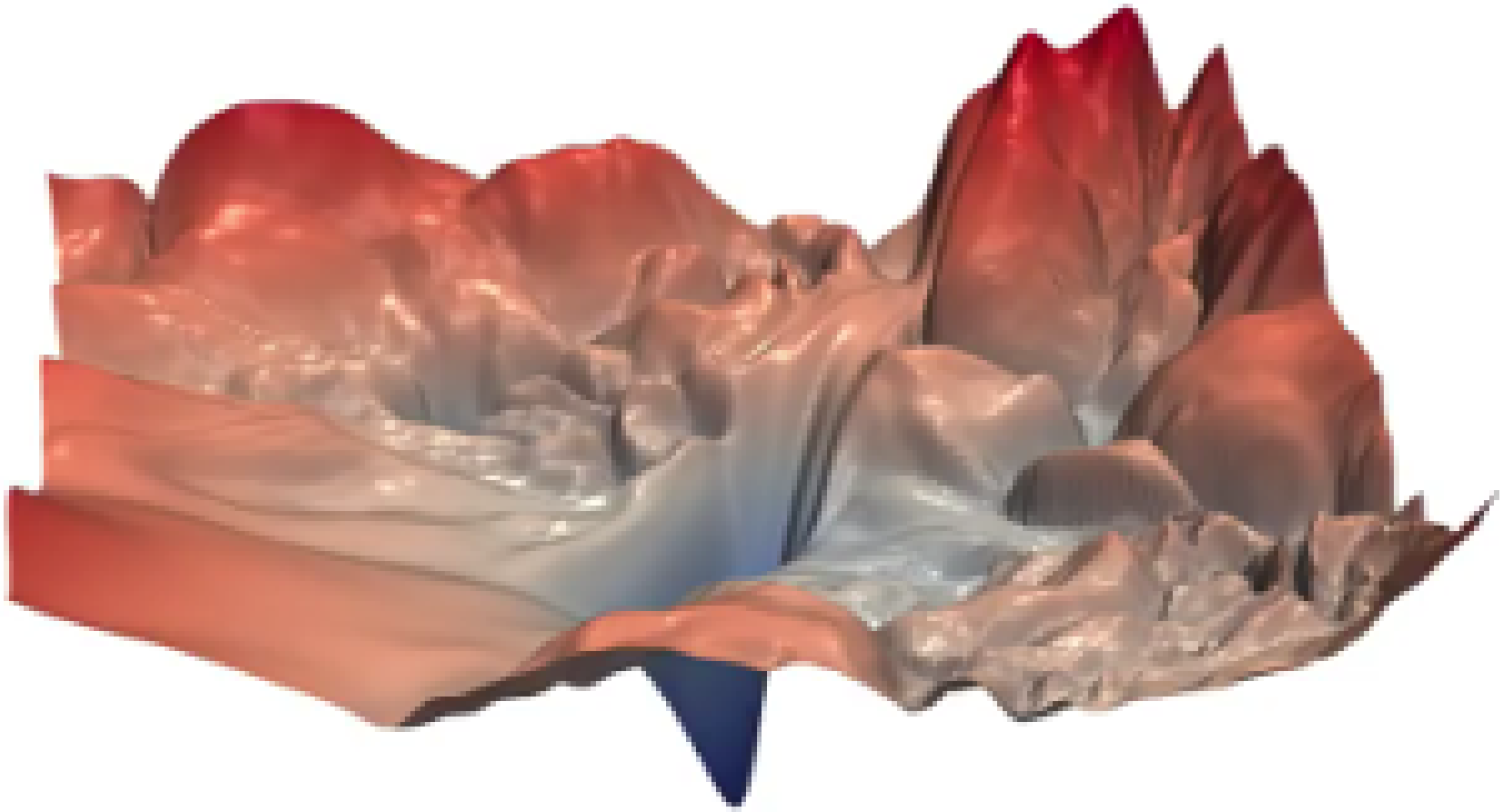
# Batch-Gradient Descent

**Algorithm 8:** Batch-Gradient Descent

1 Initialize $w$ randomly
2 **repeat**
3 | Simultaneously update all $w_j$ with $w_j - \alpha \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) x_j^{(i)}$
4 **until** *converge*;
5 **return** $w$

- At every step it evaluate all training examples
- Some time it is also called multi-variate linear regression

# Real Landscape



# Example: Gradient Descent (learning rate $\alpha$)

**Consider following data**

| | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|
| 1 | 10 | 50 | 20 | 10 |
| 2 | 11 | 31 | 22 | 12 |
| 3 | 11 | 12 | 15 | 4 |
| 4 | 20 | 55 | 20 | 22 |
| 5 | 23 | 41 | 27 | 1 |
| 6 | 31 | 12 | 35 | 9 |
| 7 | 13 | 18 | 12 | 23 |
| 8 | 21 | 55 | 16 | 16 |
| 9 | 32 | 56 | 27 | 22 |
| 10 | 8 | 22 | 35 | 11 |

# Example: Gradient Descent (learning rate $\alpha$)

**Consider following data**

| | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|
| 1 | 10 | 50 | 20 | 10 |
| 2 | 11 | 31 | 22 | 12 |
| 3 | 11 | 12 | 15 | 4 |
| 4 | 20 | 55 | 20 | 22 |
| 5 | 23 | 41 | 27 | 1 |
| 6 | 31 | 12 | 35 | 9 |
| 7 | 13 | 18 | 12 | 23 |
| 8 | 21 | 55 | 16 | 16 |
| 9 | 32 | 56 | 27 | 22 |
| 10 | 8 | 22 | 35 | 11 |

**Learning rate** $\alpha = 0.1$

# Example: Gradient Descent (learning rate $\alpha$)

**Consider following data**

| | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|
| 1 | 10 | 50 | 20 | 10 |
| 2 | 11 | 31 | 22 | 12 |
| 3 | 11 | 12 | 15 | 4 |
| 4 | 20 | 55 | 20 | 22 |
| 5 | 23 | 41 | 27 | 1 |
| 6 | 31 | 12 | 35 | 9 |
| 7 | 13 | 18 | 12 | 23 |
| 8 | 21 | 55 | 16 | 16 |
| 9 | 32 | 56 | 27 | 22 |
| 10 | 8 | 22 | 35 | 11 |

**Learning rate** $\alpha = 0.1$

J=396.662506
w=( 0.500 0.500 0.500 0.500)

J=19454472.000000
w=( -2.055 -51.070 -100.970 -62.640)

J=1036526813184.000000
w=( 590.236 11518.771 23902.906 13778.349)

J=55230041021218816.000000
w=( -135891.922 -2653678.250 -5525792.000 -3170425.000)

J=2942865354556228763648.000000
w=( 31365378.000 612476928.000 1275658624.000 731686912.000)

J=156806972273681738831495168.000000
w=( -7240111104.000 -141378551808.000 -294465732608.000
-168895037440.000)

J=835526654652697102726982742835 2.000000
w=( 1671254376448.000 32634791002112.000 67972370530304.000
38986479304704.000)

J=44520007922259187977070606888730624 0.000000

w=( -385780270759936.000 -7533178826784768.000

-15690251045437440.000 -8999357718200320.000)

# Example: Gradient Descent (learning rate $\alpha$)

**Consider following data**

| | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|
| 1 | 10 | 50 | 20 | 10 |
| 2 | 11 | 31 | 22 | 12 |
| 3 | 11 | 12 | 15 | 4 |
| 4 | 20 | 55 | 20 | 22 |
| 5 | 23 | 41 | 27 | 1 |
| 6 | 31 | 12 | 35 | 9 |
| 7 | 13 | 18 | 12 | 23 |
| 8 | 21 | 55 | 16 | 16 |
| 9 | 32 | 56 | 27 | 22 |
| 10 | 8 | 22 | 35 | 11 |

**Learning rate** $\alpha = 0.001$

# Example: Gradient Descent (learning rate $\alpha$)

**Consider following data**

| | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|
| 1 | 10 | 50 | 20 | 10 |
| 2 | 11 | 31 | 22 | 12 |
| 3 | 11 | 12 | 15 | 4 |
| 4 | 20 | 55 | 20 | 22 |
| 5 | 23 | 41 | 27 | 1 |
| 6 | 31 | 12 | 35 | 9 |
| 7 | 13 | 18 | 12 | 23 |
| 8 | 21 | 55 | 16 | 16 |
| 9 | 32 | 56 | 27 | 22 |
| 10 | 8 | 22 | 35 | 11 |

**Learning rate** $\alpha = 0.001$

| J | w |
|---|---|
| 396.663 | ( 0.500 0.500 0.500 0.500) |
| 664.137 | ( 0.474 -0.016 -0.515 -0.131) |
| 1131.021 | ( 0.508 0.631 0.881 0.628) |
| 1943.882 | ( 0.464 -0.249 -0.910 -0.435) |
| 3357.625 | ( 0.523 0.888 1.492 0.914) |
| 5815.401 | ( 0.446 -0.630 -1.641 -0.908) |
| 10087.491 | ( 0.549 1.356 2.518 1.456) |
| 17512.684 | ( 0.415 -1.274 -2.941 -1.693) |
| 30417.834 | ( 0.592 2.183 4.276 2.432) |
| 52847.020 | ( 0.359 -2.383 -5.221 -3.028) |
| 91828.805 | ( 0.668 3.630 7.314 4.151) |
| 159578.781 | ( 0.263 -4.302 -9.200 -5.330) |
| 277327.562 | ( 0.799 6.152 12.580 7.155) |
| 481973.594 | ( 0.093 -7.633 -16.125 -9.316) |
| 837646.250 | ( 1.025 10.537 21.725 12.387) |
| 1455801.375 | ( -0.201 -13.418 -28.168 -16.234) |
| 2530147.500 | ( 1.417 18.162 37.611 21.491) |
| 4397349.000 | ( -0.715 -23.472 -49.103 -28.249) |
| 7642525.500 | ( 2.097 31.415 65.218 37.319) |
| 13282603.000 | ( -1.608 -40.944 -85.492 -49.126) |
| 23084998.000 | ( 3.278 54.449 113.196 64.832) |
| 40121436.000 | ( -3.162 -71.310 -148.738 -85.405) |
| 69730584.000 | ( 5.329 94.483 196.578 112.653) |
| 121190936.000 | ( -5.863 -124.085 -258.660 -148.456) |
| 210628448.000 | ( 8.894 164.060 341.494 195.769) |
| 366069856.000 | ( -10.559 -215.809 -449.705 -258.035) |
| 636225152.000 | ( 15.088 284.983 593.355 340.226) |
| 1105751936.000 | ( -18.721 -375.224 -781.739 -448.479) |
| 1921783808.000 | ( 25.852 495.147 1031.086 591.291) |
| 3340036608.000 | ( -32.908 -652.287 -1358.811 -779.468) |

# Example: Gradient Descent (learning rate $\alpha$)

**Consider following data**

| | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|
| 1 | 10 | 50 | 20 | 10 |
| 2 | 11 | 31 | 22 | 12 |
| 3 | 11 | 12 | 15 | 4 |
| 4 | 20 | 55 | 20 | 22 |
| 5 | 23 | 41 | 27 | 1 |
| 6 | 31 | 12 | 35 | 9 |
| 7 | 13 | 18 | 12 | 23 |
| 8 | 21 | 55 | 16 | 16 |
| 9 | 32 | 56 | 27 | 22 |
| 10 | 8 | 22 | 35 | 11 |

**Learning rate** $\alpha = 0.0001$

# Example: Gradient Descent (learning rate $\alpha$)

**Consider following data**

| | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|
| 1 | 10 | 50 | 20 | 10 |
| 2 | 11 | 31 | 22 | 12 |
| 3 | 11 | 12 | 15 | 4 |
| 4 | 20 | 55 | 20 | 22 |
| 5 | 23 | 41 | 27 | 1 |
| 6 | 31 | 12 | 35 | 9 |
| 7 | 13 | 18 | 12 | 23 |
| 8 | 21 | 55 | 16 | 16 |
| 9 | 32 | 56 | 27 | 22 |
| 10 | 8 | 22 | 35 | 11 |

**Learning rate** $\alpha = 0.0001$

| J | w | |
|---|---|---|
| 396.663 | ( 0.500 0.500 0.500 0.500) | |
| 246.798 | ( 0.497 0.448 0.399 0.437) | |
| 158.286 | ( 0.495 0.408 0.321 0.388) | |
| 105.980 | ( 0.494 0.377 0.262 0.349) | |
| 75.041 | ( 0.493 0.353 0.218 0.319) | |
| 56.711 | ( 0.492 0.334 0.184 0.295) | |
| 45.826 | ( 0.491 0.320 0.159 0.276) | |
| 39.335 | ( 0.491 0.308 0.140 0.260) | |
| 35.439 | ( 0.490 0.299 0.126 0.248) | |
| 33.077 | ( 0.490 0.291 0.115 0.238) | |
| 31.621 | ( 0.490 0.285 0.108 0.229) | |
| 30.703 | ( 0.490 0.280 0.103 0.222) | |
| 30.104 | ( 0.490 0.276 0.099 0.216) | |
| 29.694 | ( 0.489 0.273 0.097 0.210) | |
| 29.399 | ( 0.489 0.270 0.096 0.206) | |
| 29.172 | ( 0.489 0.268 0.095 0.202) | |
| 28.987 | ( 0.489 0.266 0.096 0.198) | |
| 28.830 | ( 0.489 0.264 0.096 0.194) | |
| 28.689 | ( 0.489 0.262 0.097 0.191) | |
| 28.560 | ( 0.489 0.260 0.098 0.188) | |
| 28.439 | ( 0.489 0.259 0.099 0.185) | |
| 28.325 | ( 0.489 0.258 0.101 0.182) | |
| 28.216 | ( 0.489 0.256 0.102 0.179) | |
| 28.111 | ( 0.489 0.255 0.104 0.177) | |
| 28.011 | ( 0.489 0.254 0.105 0.174) | |
| 27.913 | ( 0.489 0.253 0.107 0.172) | |
| 27.819 | ( 0.489 0.252 0.109 0.170) | |
| 27.728 | ( 0.489 0.251 0.110 0.167) | |
| 27.555 | ( 0.490 0.249 0.114 0.163) | |
| 24.926 | ( 0.507 0.207 0.215 0.020) | **Iteration 300** |
| 24.768 | ( 0.710 0.219 0.213 0.005) | **Iteration 3000** |

# Example: Gradient Descent (Feature scaling)

**Feature scaling**

| | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|
| 1 | 0.08 | 0.86 | 0.35 | 10 |
| 2 | 0.12 | 0.43 | 0.43 | 12 |
| 3 | 0.12 | 0.00 | 0.13 | 4 |
| 4 | 0.50 | 0.98 | 0.35 | 22 |
| 5 | 0.62 | 0.66 | 0.65 | 1 |
| 6 | 0.96 | 0.00 | 1.00 | 9 |
| 7 | 0.21 | 0.14 | 0.00 | 23 |
| 8 | 0.54 | 0.98 | 0.17 | 16 |
| 9 | 1.00 | 1.00 | 0.65 | 22 |
| 10 | 0.00 | 0.23 | 1.00 | 11 |

**Learning rate** $\alpha = 0.1$

# Example: Gradient Descent (Feature scaling)

**Feature scaling**

| | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|
| 1 | 0.08 | 0.86 | 0.35 | 10 |
| 2 | 0.12 | 0.43 | 0.43 | 12 |
| 3 | 0.12 | 0.00 | 0.13 | 4 |
| 4 | 0.50 | 0.98 | 0.35 | 22 |
| 5 | 0.62 | 0.66 | 0.65 | 1 |
| 6 | 0.96 | 0.00 | 1.00 | 9 |
| 7 | 0.21 | 0.14 | 0.00 | 23 |
| 8 | 0.54 | 0.98 | 0.17 | 16 |
| 9 | 1.00 | 1.00 | 0.65 | 22 |
| 10 | 0.00 | 0.23 | 1.00 | 11 |

**Learning rate** $\alpha = 0.1$

| J | w | |
|---|---|---|
| 95.472 | ( 0.500 0.500 0.500 0.500) | |
| 73.399 | ( 1.679 1.025 1.220 0.983) | |
| 58.326 | ( 2.658 1.455 1.822 1.364) | |
| 48.020 | ( 3.470 1.808 2.326 1.663) | |
| 40.961 | ( 4.147 2.096 2.749 1.893) | |
| 36.116 | ( 4.710 2.331 3.106 2.066) | |
| 32.778 | ( 5.180 2.522 3.407 2.193) | |
| 30.468 | ( 5.574 2.677 3.662 2.283) | |
| 28.859 | ( 5.903 2.803 3.880 2.341) | |
| 27.729 | ( 6.181 2.904 4.066 2.373) | |
| 26.925 | ( 6.415 2.985 4.226 2.385) | |
| 26.344 | ( 6.613 3.049 4.364 2.379) | |
| 25.916 | ( 6.782 3.100 4.485 2.360) | |
| 25.593 | ( 6.926 3.140 4.590 2.329) | |
| 25.342 | ( 7.050 3.170 4.683 2.289) | |
| 25.141 | ( 7.158 3.193 4.766 2.241) | |
| 24.974 | ( 7.252 3.210 4.839 2.188) | |
| 24.833 | ( 7.334 3.222 4.906 2.129) | |
| **24.708** | ( 7.407 3.230 4.966 2.067) | **Iteration 18** |
| 24.596 | ( 7.472 3.234 5.021 2.003) | |
| 24.493 | ( 7.530 3.236 5.071 1.935) | |
| 24.397 | ( 7.583 3.235 5.118 1.866) | |
| 24.306 | ( 7.632 3.233 5.161 1.796) | |
| 24.219 | ( 7.677 3.229 5.202 1.725) | |
| 24.136 | ( 7.718 3.225 5.241 1.653) | |
| 24.056 | ( 7.757 3.219 5.277 1.581) | |
| 23.979 | ( 7.794 3.213 5.311 1.509) | |
| 23.903 | ( 7.830 3.206 5.344 1.436) | |
| 23.830 | ( 7.863 3.198 5.375 1.364) | |
| 23.759 | ( 7.896 3.191 5.405 1.292) | **Iteration 30** |
| 20.174 | ( 12.021 4.618 4.794 -7.329) | **Iteration 3000** |

# Similar Mechanism for Classification

Classification have predefined fixed number of labels (0 and 1 in this case)

| $x_1$ | $x_2$ | $x_3$ | *Class* |
|---|---|---|---|
| 10 | 50 | 20 | 1 |
| 11 | 31 | 22 | 1 |
| 11 | 12 | 15 | 0 |
| 20 | 55 | 20 | 0 |
| 23 | 41 | 27 | 0 |
| 31 | 12 | 35 | 1 |
| 13 | 18 | 12 | 0 |
| 21 | 55 | 16 | 1 |
| 32 | 56 | 27 | 0 |
| 8 | 22 | 35 | ?? |

What should come at the place of ??

# Logistic Regression

Moving from linear regression $y(x, w) = w_0 + w_1 x_1 + ... + w_n x_n$ to **logistic regression**

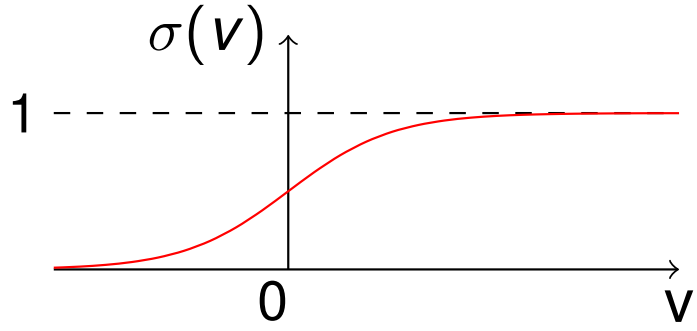
$$y(x, w) = \sigma(w_0 + w_1 x_1 + ... + w_n x_n)$$

# Logistic Regression

Moving from linear regression $y(x, w) = w_0 + w_1 x_1 + ... + w_n x_n$ to **logistic regression**

$$y(x, w) = \sigma(w_0 + w_1 x_1 + ... + w_n x_n)$$

- Enables "classification" apart from the regression. Where $\sigma$ is called as sigmoid function that produces values in range $[0, 1]$ and is defined as $\sigma(v) = \frac{1}{1+e^{-v}}$




## Decision on classification

$$classification = \begin{cases} 1 & \text{if } y(x, w) \geq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

# Logistic Regression

Moving from linear regression $y(x, w) = w_0 + w_1 x_1 + ... + w_n x_n$ to **logistic regression**

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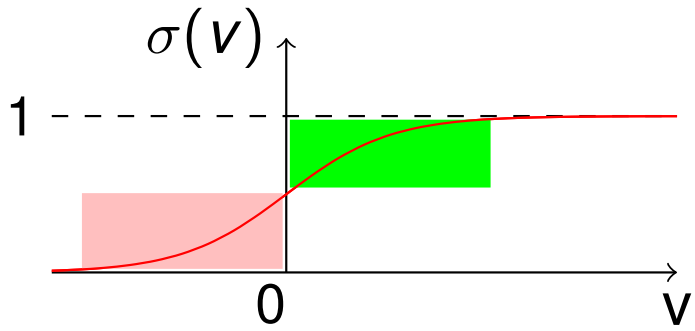



## Decision on classification

$$classification = \begin{cases} 1 & \text{if } y(x, w) \geq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

# Decision Boundary in Logistic Regression

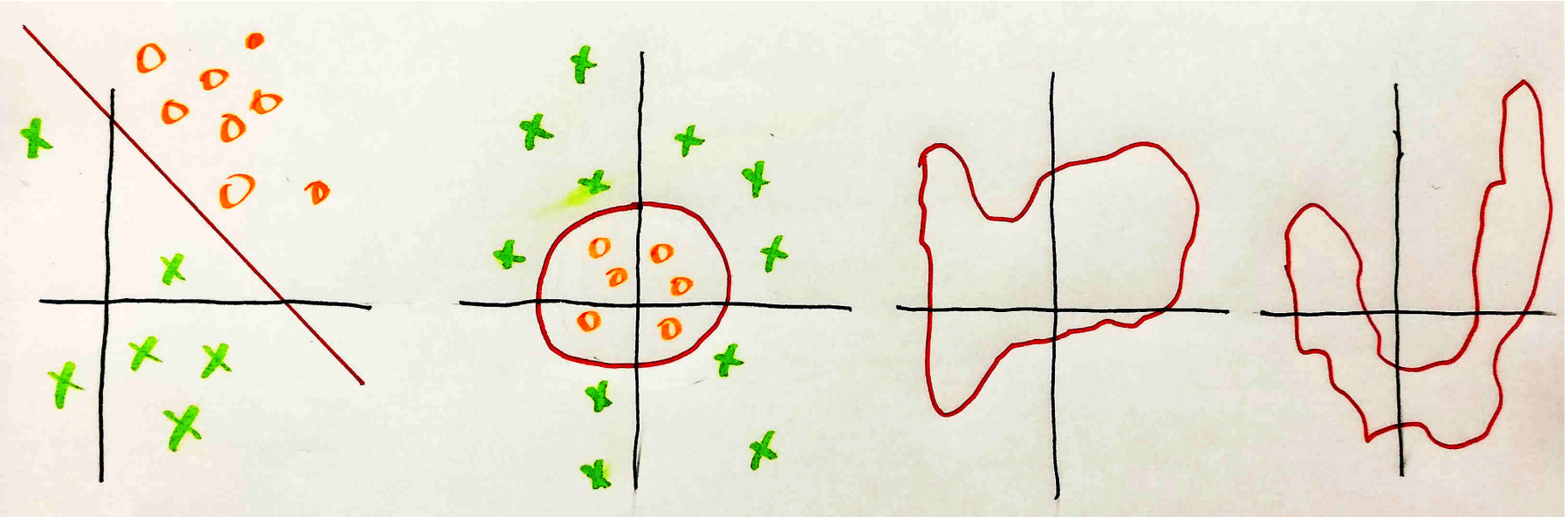
$$classification = \begin{cases} 1 & \text{if } y(x, w) \geq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

- This choice of $w$ partitions the space into two sections and the hyper-plane separating them is called **decision boundary**

# Decision Boundary in Logistic Regression

$$classification = \begin{cases} 1 & \text{if } y(x, w) \geq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

- This choice of $w$ partitions the space into two sections and the hyper-plane separating them is called **decision boundary**
- By adding more complex or polynomial terms one can get more complex decision boundary



# Cost Function

- Cost function used for the liner regression

$$J(w) = \frac{1}{2m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)})^2$$

becomes a non convex function in case of logistic regression

# Cost Function

- Cost function used for the liner regression

$$J(w) = \frac{1}{2m}\sum_{i=1}^{m}(y(x^{(i)}, w) - y^{(i)})^2$$

becomes a non convex function in case of logistic regression

## Therefore, a different cost function is chosen

$$J(w) = \frac{1}{m}\sum_{i=1}^{m} Cost(y(x^{(i)}, w), y^{(i)})$$

where

$$Cost(y(x^{(i)}, w), y^{(i)}) = \begin{cases} -\log(y(x^{(i)}, w)) & \text{if } y^{(i)} = 1 \\ -\log(1 - y(x^{(i)}, w)) & \text{otherwise} \end{cases}$$

A simplified version of this cost function is

$$Cost(y(x^{(i)}, w), y^{(i)}) = -y^{(i)}\log(y(x^{(i)}, w)) - (1 - y^{(i)})\log(1 - y(x^{(i)}, w))$$

# Learning With This Cost Function

- Learning corresponds to the minimization of $J(w)$ by changing $w$

$$\underset{w}{\operatorname{argmin}}\, J(w) = \frac{1}{m} \sum_{i=1}^{m} Cost(y(x^{(i)}, w), y^{(i)})$$

# Learning With This Cost Function

- Learning corresponds to the minimization of $J(w)$ by changing $w$

$$\underset{w}{\operatorname{argmin}}\, J(w) = \frac{1}{m}\sum_{i=1}^{m} Cost(y(x^{(i)}, w), y^{(i)})$$

$$\underset{w}{\operatorname{argmin}}\, J(w) = \frac{1}{m}\sum_{i=1}^{m}[-y^{(i)}\log(y(x^{(i)}, w)) - (1 - y^{(i)})\log(1 - y(x^{(i)}, w))]$$

# Learning With This Cost Function

- Learning corresponds to the minimization of $J(w)$ by changing $w$

$$\underset{w}{\text{argmin}}\, J(w) = \frac{1}{m}\sum_{i=1}^{m} Cost(y(x^{(i)}, w), y^{(i)})$$

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- Gradient Descent can be used for this purpose

**Algorithm 11:** Logistic Regression

1 Initialize $w$ randomly
2 **repeat**
3 | Simultaneously update all $w_j$ with $w_j - \alpha\frac{\partial}{\partial w_j}J(w)$
4 **until** *converge*;
5 **return** $w$

# The Partial Derivative Term

Recall differentiation

$$\frac{d}{dx} x^{-1} = \frac{-1}{x^2}$$

$$\frac{d}{dx} \log x = \frac{1}{x}$$

$$\frac{d}{dx} \log \sin x = \frac{1}{\sin x} \frac{d}{dx} \sin x = \frac{1}{\sin x} \cos x$$

# The Partial Derivative Term

Recall differentiation

$$\frac{d}{dx}x^{-1} = \frac{-1}{x^2} \qquad \frac{d}{dx}\log x = \frac{1}{x} \qquad \frac{d}{dx}\log\sin x = \frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{1}{\sin x}\cos x$$

## Let $v = w_0x_0 + w_1x_1 + \ldots + w_nx_n$ Then

$$\frac{\partial}{\partial w_j}v =$$

# The Partial Derivative Term

Recall differentiation

$$\frac{d}{dx}x^{-1} = \frac{-1}{x^2} \qquad \frac{d}{dx}\log x = \frac{1}{x} \qquad \frac{d}{dx}\log\sin x = \frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{1}{\sin x}\cos x$$

## Let $v = w_0x_0 + w_1x_1 + \ldots + w_nx_n$ Then

$$\frac{\partial}{\partial w_j}v = \frac{\partial}{\partial w_j}(w_0x_0 + w_1x_1 + \ldots + w_nx_n) =$$

# The Partial Derivative Term

Recall differentiation

$$\frac{d}{dx}x^{-1} = \frac{-1}{x^2} \qquad \frac{d}{dx}\log x = \frac{1}{x} \qquad \frac{d}{dx}\log\sin x = \frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{1}{\sin x}\cos x$$

## Let $v = w_0x_0 + w_1x_1 + \ldots + w_nx_n$ Then

$$\frac{\partial}{\partial w_j}v = \frac{\partial}{\partial w_j}(w_0x_0 + w_1x_1 + \ldots + w_nx_n) = x_j$$

# The Partial Derivative Term

Recall differentiation

$$\frac{d}{dx}x^{-1} = \frac{-1}{x^2} \qquad \frac{d}{dx}\log x = \frac{1}{x} \qquad \frac{d}{dx}\log\sin x = \frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{1}{\sin x}\cos x$$

## Let $v = w_0x_0 + w_1x_1 + \ldots + w_nx_n$ Then

$$\frac{\partial}{\partial w_j}v = \frac{\partial}{\partial w_j}(w_0x_0 + w_1x_1 + \ldots + w_nx_n) = x_j$$

$$J(w) = \frac{1}{m}\sum_{i=1}^{m}[-y^{(i)}\log(y(x^{(i)}, w)) - (1 - y^{(i)})\log(1 - y(x^{(i)}, w))]$$

# The Partial Derivative Term

Recall differentiation

$$\frac{d}{dx}x^{-1} = \frac{-1}{x^2} \qquad \frac{d}{dx}\log x = \frac{1}{x} \qquad \frac{d}{dx}\log\sin x = \frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{1}{\sin x}\cos x$$

## Let $v = w_0x_0 + w_1x_1 + \ldots + w_nx_n$ Then

$$\frac{\partial}{\partial w_j}v = \frac{\partial}{\partial w_j}(w_0x_0 + w_1x_1 + \ldots + w_nx_n) = x_j$$

$$\begin{aligned} J(w) &= \frac{1}{m}\sum_{i=1}^{m}[-y^{(i)}\log(y(x^{(i)}, w)) - (1 - y^{(i)})\log(1 - y(x^{(i)}, w))] \\ \frac{\partial}{\partial w_j}J(w) &= \frac{1}{m}\sum_{i=1}^{m}[-\frac{\partial}{\partial w_j}y^{(i)}\log(y(x^{(i)}, w)) - \frac{\partial}{\partial w_j}(1 - y^{(i)})\log(1 - y(x^{(i)}, w))] \end{aligned}$$

# The Partial Derivative Term

Recall differentiation

$$\frac{d}{dx}x^{-1} = \frac{-1}{x^2} \qquad \frac{d}{dx}\log x = \frac{1}{x} \qquad \frac{d}{dx}\log\sin x = \frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{1}{\sin x}\cos x$$

## Let $v = w_0x_0 + w_1x_1 + \ldots + w_nx_n$ Then

$$\frac{\partial}{\partial w_j}v = \frac{\partial}{\partial w_j}(w_0x_0 + w_1x_1 + \ldots + w_nx_n) = x_j$$

$$\begin{aligned}
J(w) &= \frac{1}{m}\sum_{i=1}^{m}[-y^{(i)}\log(y(x^{(i)}, w)) - (1 - y^{(i)})\log(1 - y(x^{(i)}, w))] \\
\frac{\partial}{\partial w_j}J(w) &= \frac{1}{m}\sum_{i=1}^{m}[-\frac{\partial}{\partial w_j}y^{(i)}\log(y(x^{(i)}, w)) - \frac{\partial}{\partial w_j}(1 - y^{(i)})\log(1 - y(x^{(i)}, w))] \\
&= \frac{1}{m}\sum_{i=1}^{m}[-A - B] \qquad (3)
\end{aligned}$$

# The Partial Derivative

$$A = \frac{\partial}{\partial w_j} y^{(i)} \log(y(x^{(i)}, w))$$

# The Partial Derivative

$$
\begin{aligned}
A &= \frac{\partial}{\partial w_j} y^{(i)} \log(y(x^{(i)}, w)) \\
&= y^{(i)} \times \frac{\partial}{\partial w_j} \log(y(x^{(i)}, w))
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
A &= \frac{\partial}{\partial w_j} y^{(i)} \log(y(x^{(i)}, w)) \\
&= y^{(i)} \times \frac{\partial}{\partial w_j} \log(y(x^{(i)}, w)) \\
&= y^{(i)} \times \frac{1}{y(x^{(i)}, w)} \times \frac{\partial}{\partial w_j} y(x^{(i)}, w)
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
A &= \frac{\partial}{\partial w_j} y^{(i)} \log(y(x^{(i)}, w)) \\
&= y^{(i)} \times \frac{\partial}{\partial w_j} \log(y(x^{(i)}, w)) \\
&= y^{(i)} \times \frac{1}{y(x^{(i)}, w)} \times \frac{\partial}{\partial w_j} y(x^{(i)}, w) \\
&= y^{(i)} \times \frac{1}{\frac{1}{1+e^{-v}}} \times \frac{\partial}{\partial w_j} \frac{1}{1+e^{-v}}
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
A &= \frac{\partial}{\partial w_j} y^{(i)} \log(y(x^{(i)}, w)) \\
&= y^{(i)} \times \frac{\partial}{\partial w_j} \log(y(x^{(i)}, w)) \\
&= y^{(i)} \times \frac{1}{y(x^{(i)}, w)} \times \frac{\partial}{\partial w_j} y(x^{(i)}, w) \\
&= y^{(i)} \times \frac{1}{\frac{1}{1+e^{-v}}} \times \frac{\partial}{\partial w_j} \frac{1}{1+e^{-v}} \\
&= y^{(i)} \times (1+e^{-v}) \times \frac{-1}{(1+e^{-v})^2} \times \frac{\partial}{\partial w_j} (1+e^{-v})
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
A &= \frac{\partial}{\partial w_j} y^{(i)} \log(y(x^{(i)}, w)) \\
&= y^{(i)} \times \frac{\partial}{\partial w_j} \log(y(x^{(i)}, w)) \\
&= y^{(i)} \times \frac{1}{y(x^{(i)}, w)} \times \frac{\partial}{\partial w_j} y(x^{(i)}, w) \\
&= y^{(i)} \times \frac{1}{\frac{1}{1+e^{-v}}} \times \frac{\partial}{\partial w_j} \frac{1}{1+e^{-v}} \\
&= y^{(i)} \times (1+e^{-v}) \times \frac{-1}{(1+e^{-v})^2} \times \frac{\partial}{\partial w_j} (1+e^{-v}) \\
&= \frac{-y^{(i)}}{1+e^{-v}} \times (0 + e^{-v} \times (-1) \frac{\partial}{\partial w_j} v)
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
A &= \frac{\partial}{\partial w_j} y^{(i)} \log(y(x^{(i)}, w)) \\
&= y^{(i)} \times \frac{\partial}{\partial w_j} \log(y(x^{(i)}, w)) \\
&= y^{(i)} \times \frac{1}{y(x^{(i)}, w)} \times \frac{\partial}{\partial w_j} y(x^{(i)}, w) \\
&= y^{(i)} \times \frac{1}{\frac{1}{1+e^{-v}}} \times \frac{\partial}{\partial w_j} \frac{1}{1+e^{-v}} \\
&= y^{(i)} \times (1+e^{-v}) \times \frac{-1}{(1+e^{-v})^2} \times \frac{\partial}{\partial w_j} (1+e^{-v}) \\
&= \frac{-y^{(i)}}{1+e^{-v}} \times (0 + e^{-v} \times (-1) \frac{\partial}{\partial w_j} v) \\
&= y^{(i)} \times \frac{e^{-v}}{1+e^{-v}} \times x_j \qquad (4)
\end{aligned}
$$

# The Partial Derivative

$$B = \frac{\partial}{\partial w_j}(1 - y^{(i)}) \log(1 - y(x^{(i)}, w))$$

# The Partial Derivative

$$
\begin{aligned}
B &= \frac{\partial}{\partial w_j}(1 - y^{(i)}) \log(1 - y(x^{(i)}, w)) \\
&= (1 - y^{(i)}) \times \frac{1}{1 - y(x^{(i)}, w)} \times \frac{\partial}{\partial w_j}(1 - y(x^{(i)}, w))
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
B &= \frac{\partial}{\partial w_j}(1 - y^{(i)}) \log(1 - y(x^{(i)}, w)) \\
&= (1 - y^{(i)}) \times \frac{1}{1 - y(x^{(i)}, w)} \times \frac{\partial}{\partial w_j}(1 - y(x^{(i)}, w)) \\
&= (1 - y^{(i)}) \times \frac{-1}{1 - \frac{1}{1+e^{-v}}} \times \frac{\partial}{\partial w_j} y(x^{(i)}, w)
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
B &= \frac{\partial}{\partial w_j}(1 - y^{(i)}) \log(1 - y(x^{(i)}, w)) \\
&= (1 - y^{(i)}) \times \frac{1}{1 - y(x^{(i)}, w)} \times \frac{\partial}{\partial w_j}(1 - y(x^{(i)}, w)) \\
&= (1 - y^{(i)}) \times \frac{-1}{1 - \frac{1}{1+e^{-v}}} \times \frac{\partial}{\partial w_j} y(x^{(i)}, w) \\
&= (1 - y^{(i)}) \times \frac{(-1)(1 + e^{-v})}{e^{-v}} \times \frac{\partial}{\partial w_j} \frac{1}{1 + e^{-v}}
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
B &= \frac{\partial}{\partial w_j}(1 - y^{(i)}) \log(1 - y(x^{(i)}, w)) \\
&= (1 - y^{(i)}) \times \frac{1}{1 - y(x^{(i)}, w)} \times \frac{\partial}{\partial w_j}(1 - y(x^{(i)}, w)) \\
&= (1 - y^{(i)}) \times \frac{-1}{1 - \frac{1}{1+e^{-v}}} \times \frac{\partial}{\partial w_j} y(x^{(i)}, w) \\
&= (1 - y^{(i)}) \times \frac{(-1)(1 + e^{-v})}{e^{-v}} \times \frac{\partial}{\partial w_j} \frac{1}{1 + e^{-v}} \\
&= (1 - y^{(i)}) \times \frac{(-1)(1 + e^{-v})}{e^{-v}} \times \frac{-1}{(1 + e^{-v})^2} \times \frac{\partial}{\partial w_j}(1 + e^{-v})
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
B &= \frac{\partial}{\partial w_j}(1 - y^{(i)}) \log(1 - y(x^{(i)}, w)) \\
&= (1 - y^{(i)}) \times \frac{1}{1 - y(x^{(i)}, w)} \times \frac{\partial}{\partial w_j}(1 - y(x^{(i)}, w)) \\
&= (1 - y^{(i)}) \times \frac{-1}{1 - \frac{1}{1+e^{-v}}} \times \frac{\partial}{\partial w_j} y(x^{(i)}, w) \\
&= (1 - y^{(i)}) \times \frac{(-1)(1 + e^{-v})}{e^{-v}} \times \frac{\partial}{\partial w_j} \frac{1}{1 + e^{-v}} \\
&= (1 - y^{(i)}) \times \frac{(-1)(1 + e^{-v})}{e^{-v}} \times \frac{-1}{(1 + e^{-v})^2} \times \frac{\partial}{\partial w_j}(1 + e^{-v}) \\
&= (1 - y^{(i)}) \times \frac{(-1)(1 + e^{-v})}{e^{-v}} \times \frac{-1}{(1 + e^{-v})^2} \times (0 + e^{-v} \frac{\partial}{\partial w_j}(-v))
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
B &= \frac{\partial}{\partial w_j}(1 - y^{(i)}) \log(1 - y(x^{(i)}, w)) \\
&= (1 - y^{(i)}) \times \frac{1}{1 - y(x^{(i)}, w)} \times \frac{\partial}{\partial w_j}(1 - y(x^{(i)}, w)) \\
&= (1 - y^{(i)}) \times \frac{-1}{1 - \frac{1}{1+e^{-v}}} \times \frac{\partial}{\partial w_j} y(x^{(i)}, w) \\
&= (1 - y^{(i)}) \times \frac{(-1)(1 + e^{-v})}{e^{-v}} \times \frac{\partial}{\partial w_j} \frac{1}{1 + e^{-v}} \\
&= (1 - y^{(i)}) \times \frac{(-1)(1 + e^{-v})}{e^{-v}} \times \frac{-1}{(1 + e^{-v})^2} \times \frac{\partial}{\partial w_j}(1 + e^{-v}) \\
&= (1 - y^{(i)}) \times \frac{(-1)(1 + e^{-v})}{e^{-v}} \times \frac{-1}{(1 + e^{-v})^2} \times (0 + e^{-v} \frac{\partial}{\partial w_j}(-v)) \\
&= (1 - y^{(i)}) \times \frac{(-1)(1 + e^{-v})}{e^{-v}} \times \frac{e^{-v}}{(1 + e^{-v})^2} \times \frac{\partial}{\partial w_j} v
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
B &= \frac{\partial}{\partial w_j}(1-y^{(i)})\log(1-y(x^{(i)},w)) \\
&= (1-y^{(i)})\times\frac{1}{1-y(x^{(i)},w)}\times\frac{\partial}{\partial w_j}(1-y(x^{(i)},w)) \\
&= (1-y^{(i)})\times\frac{-1}{1-\frac{1}{1+e^{-v}}}\times\frac{\partial}{\partial w_j}y(x^{(i)},w) \\
&= (1-y^{(i)})\times\frac{(-1)(1+e^{-v})}{e^{-v}}\times\frac{\partial}{\partial w_j}\frac{1}{1+e^{-v}} \\
&= (1-y^{(i)})\times\frac{(-1)(1+e^{-v})}{e^{-v}}\times\frac{-1}{(1+e^{-v})^2}\times\frac{\partial}{\partial w_j}(1+e^{-v}) \\
&= (1-y^{(i)})\times\frac{(-1)(1+e^{-v})}{e^{-v}}\times\frac{-1}{(1+e^{-v})^2}\times(0+e^{-v}\frac{\partial}{\partial w_j}(-v)) \\
&= (1-y^{(i)})\times\frac{(-1)(1+e^{-v})}{e^{-v}}\times\frac{e^{-v}}{(1+e^{-v})^2}\times\frac{\partial}{\partial w_j}v \\
&= (1-y^{(i)})\times\frac{-1}{1+e^{-v}}\times x_j
\end{aligned}
\tag{5}
$$

# The Partial Derivative

$$\frac{\partial}{\partial w_j} J(w) = \frac{1}{m} \sum_{i=1}^{m} [-A - B]$$

# The Partial Derivative

$$\begin{aligned}
\frac{\partial}{\partial w_j} J(w) &= \frac{1}{m} \sum_{i=1}^{m} [-A - B] \\
&= \frac{1}{m} \sum_{i=1}^{m} [-y^{(i)} \times \frac{e^{-v}}{1 + e^{-v}} \times x_j - (1 - y^{(i)}) \times \frac{-1}{1 + e^{-v}} \times x_j]
\end{aligned}$$

# The Partial Derivative

$$
\begin{aligned}
\frac{\partial}{\partial w_j} J(w) &= \frac{1}{m} \sum_{i=1}^{m} [-A - B] \\
&= \frac{1}{m} \sum_{i=1}^{m} [-y^{(i)} \times \frac{e^{-v}}{1 + e^{-v}} \times x_j - (1 - y^{(i)}) \times \frac{-1}{1 + e^{-v}} \times x_j] \\
&= \frac{1}{m} \sum_{i=1}^{m} [(1 - y^{(i)}) - y^{(i)} \times e^{-v}] \times \frac{x_j}{1 + e^{-v}}
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
\frac{\partial}{\partial w_j} J(w) &= \frac{1}{m} \sum_{i=1}^{m} [-A - B] \\
&= \frac{1}{m} \sum_{i=1}^{m} [-y^{(i)} \times \frac{e^{-v}}{1 + e^{-v}} \times x_j - (1 - y^{(i)}) \times \frac{-1}{1 + e^{-v}} \times x_j] \\
&= \frac{1}{m} \sum_{i=1}^{m} [(1 - y^{(i)}) - y^{(i)} \times e^{-v}] \times \frac{x_j}{1 + e^{-v}} \\
&= \frac{1}{m} \sum_{i=1}^{m} [1 - y^{(i)} \times (1 + e^{-v})] \times \frac{x_j}{1 + e^{-v}}
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
\frac{\partial}{\partial w_j} J(w) &= \frac{1}{m}\sum_{i=1}^{m}[-A - B] \\
&= \frac{1}{m}\sum_{i=1}^{m}[-y^{(i)} \times \frac{e^{-v}}{1+e^{-v}} \times x_j - (1-y^{(i)}) \times \frac{-1}{1+e^{-v}} \times x_j] \\
&= \frac{1}{m}\sum_{i=1}^{m}[(1-y^{(i)}) - y^{(i)} \times e^{-v}] \times \frac{x_j}{1+e^{-v}} \\
&= \frac{1}{m}\sum_{i=1}^{m}[1 - y^{(i)} \times (1+e^{-v})] \times \frac{x_j}{1+e^{-v}} \\
&= \frac{1}{m}\sum_{i=1}^{m}[\frac{1}{1+e^{-v}} - y^{(i)}] \times x_j
\end{aligned}
$$

# The Partial Derivative

$$
\begin{aligned}
\frac{\partial}{\partial w_j} J(w) &= \frac{1}{m}\sum_{i=1}^{m}[-A - B] \\
&= \frac{1}{m}\sum_{i=1}^{m}[-y^{(i)} \times \frac{e^{-v}}{1+e^{-v}} \times x_j - (1-y^{(i)}) \times \frac{-1}{1+e^{-v}} \times x_j] \\
&= \frac{1}{m}\sum_{i=1}^{m}[(1-y^{(i)}) - y^{(i)} \times e^{-v}] \times \frac{x_j}{1+e^{-v}} \\
&= \frac{1}{m}\sum_{i=1}^{m}[1 - y^{(i)} \times (1+e^{-v})] \times \frac{x_j}{1+e^{-v}} \\
&= \frac{1}{m}\sum_{i=1}^{m}[\frac{1}{1+e^{-v}} - y^{(i)}] \times x_j \\
&= \frac{1}{m}\sum_{i=1}^{m}[y(x^{(i)}, w) - y^{(i)}] \times x_j \qquad (6)
\end{aligned}
$$

# The Partial Derivative

Partial derivative term

$$\frac{\partial}{\partial w_j} J(w) = \frac{\partial}{\partial w_j} \frac{1}{m} \sum_{i=1}^{m} [-y^{(i)} \log(y(x^{(i)}, w)) - (1 - y^{(i)}) \log(1 - y(x^{(i)}, w))]$$

# The Partial Derivative

Partial derivative term

$$\frac{\partial}{\partial w_j} J(w) = \frac{\partial}{\partial w_j} \frac{1}{m} \sum_{i=1}^{m} [-y^{(i)} \log(y(x^{(i)}, w)) - (1 - y^{(i)}) \log(1 - y(x^{(i)}, w))]$$

comes out to be

$$\frac{\partial}{\partial w_j} J(w) = \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) x_j^{(i)}$$

# The Partial Derivative

Partial derivative term

$$\frac{\partial}{\partial w_j} J(w) = \frac{\partial}{\partial w_j} \frac{1}{m} \sum_{i=1}^{m} [-y^{(i)} \log(y(x^{(i)}, w)) - (1 - y^{(i)}) \log(1 - y(x^{(i)}, w))]$$

comes out to be

$$\frac{\partial}{\partial w_j} J(w) = \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) x_j^{(i)}$$

**Algorithm 14:** Logistic Regression

1. Initialize $w$ randomly
2. **repeat**
3. Simultaneously update all $w_j$ with $w_j - \alpha \times \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) x_j^{(i)}$
4. **until** *converge*;
5. **return** $w$

# The Partial Derivative

Partial derivative term

$$\frac{\partial}{\partial w_j} J(w) = \frac{\partial}{\partial w_j} \frac{1}{m} \sum_{i=1}^{m} [-y^{(i)} \log(y(x^{(i)}, w)) - (1 - y^{(i)}) \log(1 - y(x^{(i)}, w))]$$

comes out to be

$$\frac{\partial}{\partial w_j} J(w) = \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) x_j^{(i)}$$

**Algorithm 15:** Logistic Regression

1. Initialize $w$ randomly
2. **repeat**
3. Simultaneously update all $w_j$ with $w_j - \alpha \times \frac{1}{m} \sum_{i=1}^{m} (y(x^{(i)}, w) - y^{(i)}) x_j^{(i)}$
4. **until** *converge*;
5. **return** $w$

It looks identical to liner regression but, $y(x^{(i)}, w)$ is different here $y(x^{(i)}, w) = \frac{1}{1 + e^{-(w_0 + w_1 x_1^{(i)} + \ldots + w_n x_n^{(i)})}}$

# Example: Logistic Regression

**Consider following data**

| | $x_1$ | $x_2$ | $x_3$ | *Class* |
|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 1 |
| 2 | 3 | 2 | 2 | 1 |
| 3 | 2 | 3 | 2 | 1 |
| 4 | 2 | 2 | 3 | 1 |
| 5 | 7 | 6 | 9 | 0 |
| 6 | 9 | 7 | 6 | 0 |
| 7 | 9 | 6 | 7 | 0 |
| 8 | 6 | 8 | 9 | 0 |
| 9 | 8 | 9 | 6 | 0 |
| 10 | 8 | 9 | 9 | 0 |

**Learning rate** $\alpha = 0.01$

# Example: Logistic Regression

**Consider following data**

| | $x_1$ | $x_2$ | $x_3$ | *Class* |
|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 1 |
| 2 | 3 | 2 | 2 | 1 |
| 3 | 2 | 3 | 2 | 1 |
| 4 | 2 | 2 | 3 | 1 |
| 5 | 7 | 6 | 9 | 0 |
| 6 | 9 | 7 | 6 | 0 |
| 7 | 9 | 6 | 7 | 0 |
| 8 | 6 | 8 | 9 | 0 |
| 9 | 8 | 9 | 6 | 0 |
| 10 | 8 | 9 | 9 | 0 |

**Learning rate** $\alpha = 0.01$

| J | w |
|---|---|
| 6.912 | ( 0.500 0.500 0.500 0.500) |
| 6.496 | ( 0.494 0.453 0.455 0.454) |
| 5.944 | ( 0.488 0.406 0.410 0.408) |
| 5.316 | ( 0.482 0.360 0.366 0.363) |
| 4.692 | ( 0.477 0.313 0.321 0.317) |
| 4.072 | ( 0.471 0.267 0.277 0.272) |
| 3.460 | ( 0.465 0.221 0.233 0.227) |
| 2.860 | ( 0.460 0.175 0.189 0.182) |
| 2.279 | ( 0.454 0.130 0.146 0.138) |
| 1.735 | ( 0.449 0.086 0.104 0.095) |
| 1.262 | ( 0.445 0.044 0.064 0.054) |
| 0.906 | ( 0.441 0.008 0.029 0.018) |
| 0.685 | ( 0.438 -0.022 0.000 -0.011) |
| 0.566 | ( 0.437 -0.044 -0.020 -0.032) |
| 0.504 | ( 0.436 -0.060 -0.035 -0.048) |
| 0.470 | ( 0.436 -0.072 -0.046 -0.059) |
| 0.451 | ( 0.436 -0.081 -0.055 -0.068) |
| 0.438 | ( 0.436 -0.088 -0.061 -0.074) |
| 0.431 | ( 0.437 -0.093 -0.066 -0.080) |
| 0.425 | ( 0.438 -0.098 -0.070 -0.084) |
| 0.422 | ( 0.439 -0.101 -0.074 -0.088) |
| 0.419 | ( 0.440 -0.105 -0.077 -0.091) |
| 0.417 | ( 0.441 -0.107 -0.079 -0.093) |
| 0.416 | ( 0.443 -0.110 -0.081 -0.095) |
| 0.415 | ( 0.444 -0.112 -0.082 -0.097) **Iteration 25** |
| 0.412 | ( 0.451 -0.119 -0.088 -0.103) **Iteration 30** |
| 0.348 | ( 0.857 -0.179 -0.084 -0.132) **Iteration 300** |
| 0.116 | ( 3.256 -0.409 -0.135 -0.291) **Iteration 3000** |
| 0.012 | ( 7.596 -0.748 -0.361 -0.588) **Iteration 30000** |
| 0.001 | ( 11.975 -1.091 -0.599 -0.896) **Iteration 300000** |

# Example: Find J

As $(w_0, w_1, w_2, w_3)$=$(0.5, 0.5, 0.5, 0.5)$, $v = w_0 + w_1 x_1 + w_2 x_2 + w_3 x_3$
$y(x^{(i)}, w) = \sigma(v)$
And log term is $-y^{(i)} \log(y(x^{(i)}, w)) - (1 - y^{(i)}) \log(1 - y(x^{(i)}, w))$

| $i$ | $x_1$ | $x_2$ | $x_3$ | $y^{(i)}$ | v | $y(x^{(i)}, w)$ | log term |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 1 | 3.5 | 0.970 | 0.029 |
| 2 | 3 | 2 | 2 | 1 | 4.0 | 0.982 | 0.018 |
| 3 | 2 | 3 | 2 | 1 | 4.0 | 0.982 | 0.018 |
| 4 | 2 | 2 | 3 | 1 | 4.0 | 0.982 | 0.018 |
| 5 | 7 | 6 | 9 | 0 | 11.5 | 0.999 | 11.49 |
| 6 | 9 | 7 | 6 | 0 | 11.5 | 0.999 | 11.49 |
| 7 | 9 | 6 | 7 | 0 | 11.5 | 0.999 | 11.49 |
| 8 | 6 | 8 | 9 | 0 | 12 | 0.999 | 11.51 |
| 9 | 8 | 9 | 6 | 0 | 12 | 0.999 | 11.51 |
| 10 | 8 | 9 | 9 | 0 | 13 | 0.999 | 11.51 |
| | | | | | | Total/10: | 6.9118 |

# Example: Find next W

Let $(w_0, w_1, w_2, w_3)$=(0.5, 0.5, 0.5, 0.5) and $t_i = (y(x^{(i)}, w) - y^{(i)})x_j^{(i)}$

Then $\frac{1}{m}\sum_{i=1}^{m}(y(x^{(i)}, w) - y^{(i)})x_j^{(i)} = \frac{1}{m}\sum_{i=1}^{m} t_i$ let $\hat{y}^{(i)} = y(x^{(i)}, w)$

Then update $w_j$ with $w_j - \alpha \times \frac{1}{m}\sum_{i=1}^{m} t_i$ we have set $\alpha = 0.01$

| $i$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $y^{(i)}$ | $\hat{y}^{(i)}$ | $t_0$ | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 2 | 1 | 0.970 | -0.029 | -0.058 | -0.058 | -0.058 |
| 2 | 1 | 3 | 2 | 2 | 1 | 0.982 | -0.017 | -0.053 | -0.035 | -0.035 |
| 3 | 1 | 2 | 3 | 2 | 1 | 0.982 | -0.017 | -0.035 | -0.053 | -0.035 |
| 4 | 1 | 2 | 2 | 3 | 1 | 0.982 | -0.017 | -0.035 | -0.035 | -0.053 |
| 5 | 1 | 7 | 6 | 9 | 0 | 0.999 | 0.999 | 6.999 | 5.999 | 8.999 |
| 6 | 1 | 9 | 7 | 6 | 0 | 0.999 | 0.999 | 8.999 | 6.999 | 5.999 |
| 7 | 1 | 9 | 6 | 7 | 0 | 0.999 | 0.999 | 8.999 | 5.999 | 6.999 |
| 8 | 1 | 6 | 8 | 9 | 0 | 0.999 | 0.999 | 5.999 | 7.999 | 8.999 |
| 9 | 1 | 8 | 9 | 6 | 0 | 0.999 | 0.999 | 7.999 | 8.999 | 5.999 |
| 10 | 1 | 8 | 9 | 9 | 0 | 0.999 | 0.999 | 7.999 | 8.999 | 8.999 |
| | | | | | | Total | 5.916 | 46.815 | 44.815 | 45.815 |
| | | | | | | $w_i - \alpha \times (total/m)$ | 0.494 | 0.453 | 0.455 | 0.454 |

# Example: Classification across Iterations

Following table shows classification as the weights get modified along $1^{st}$, $100^{th}$, $300^{th}$ and $500^{th}$ iteration

| $i$ | $x_1$ | $x_2$ | $x_3$ | $y^{(i)}$ | 1 | 100 | 300 | 500 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 1 | 1 | 0 | 1 | 1 |
| 2 | 3 | 2 | 2 | 1 | 1 | 0 | 0 | 1 |
| 3 | 2 | 3 | 2 | 1 | 1 | 0 | 1 | 1 |
| 4 | 2 | 2 | 3 | 1 | 1 | 0 | 1 | 1 |
| 5 | 7 | 6 | 9 | 0 | 1 | 0 | 0 | 0 |
| 6 | 9 | 7 | 6 | 0 | 1 | 0 | 0 | 0 |
| 7 | 9 | 6 | 7 | 0 | 1 | 0 | 0 | 0 |
| 8 | 6 | 8 | 9 | 0 | 1 | 0 | 0 | 0 |
| 9 | 8 | 9 | 6 | 0 | 1 | 0 | 0 | 0 |
| 10 | 8 | 9 | 9 | 0 | 1 | 0 | 0 | 0 |

# Thank You!

**Thank you very much for your attention!**

**Queries ?**

(Reference[1])

[1] 1) Book - *AIMA*, ch-14, Russell and Norvig. 2) Book - *Bayeian Reasoning and Machine Learning*, ch-04, David Barber.