Tutorial 5, Design and Analysis of Algorithms, 2021
Use *Dynamic Programming* to design the algorithms. Study section 15.1 (Rod Cutting) from the textbook to answer questions 7–9.

1. Find *all possible* shortest salesman tours and cost for the following instance of the *Traveling Salseman Problem* for a directed K_4 graph having the following adjacency matrix:
$$\begin{bmatrix} 0 & 8 & 3 & 6 \\ 2 & 0 & 1 & 5 \\ 4 & 7 & 0 & 9 \\ 8 & 2 & 5 & 0 \end{bmatrix}.$$

2. Determine a *Longest Common Subsequence* of $< 1,0,0,1,0,1,0,1,1,0 >$ and $< 0,1,0,1,1,0,1,1,0,1,0 >$ using the *Dynamic Programming Algorithm.*

3. Solve the following instance of the *Matrix Chain Multiplication Problem:*
$< A_{1\times2}, B_{2\times3}, C_{3\times4}, D_{4\times5}, E_{5\times6}, F_{6\times7} >$.

4. Prove that the number of different binary trees with n nodes is
$$\frac{1}{n+1}\binom{2n}{n}.$$

5. A palindrome is a nonempty string over some alphabet that reads the same forward and backward. Examples of palindromes are all strings of length 1, `civic`, `racecar`, and `aibohphobia` (fear of palindromes). Give an efficient algorithm to find the longest palindrome that is a subsequence of a given input string. For example, given the input `character`, your algorithm should return `carac`. What is the running time of your algorithm? Show the working of your algorithm on input `character`.

6. The owners of an independently operated gas station are faced with the following situation. They have a large underground tank in which they store gas; the tank can hold up to L gallons at one time. Ordering gas is quite expensive, so they want to order relatively rarely. For each order, they need to pay a fixed price P for delivery in addition to the cost of the gas ordered. However, it costs c to store a gallon of gas for an extra day (l gallons remaining at the end of a day will give extra cost of lc), so ordering too much ahead increases the storage cost. They are planning to close for a week in the winter, and they want their tank to be empty by the time they close. Luckily, based on years of experience, they have accurate projections for how much gas they will need each day until this point in time. Assume that there are n days left until they close, and they need g_i ($g_i \leq L$) gallons of gas for each of the days $i = 1, \ldots, n$. Assume that the tank is empty at the end of day 0. Design an efficient algorithm to decide on which days they should place orders, and how much to order so as to minimize their total cost. Also find the time complexity of the algorithm.

7. Show, by means of a counterexample, that the following "greedy" strategy does not always determine an optimal way to cut rods. Define the *density* of a rod of length i to be $\frac{p_i}{i}$, that is, its value per inch. The greedy strategy for a rod of length n cuts off a first piece of length i, where $1 \leq i \leq n$, having maximum density. It then continues by applying the greedy strategy to the remaining piece of length $n - i$.

8. Consider a modification of the rod-cutting problem in which, in addition to a price p_i for each rod, each cut incurs a fixed cost of c. The revenue associated with a solution is now the sum of the prices of the pieces minus the costs of making the cuts. Give a dynamic-programming algorithm to solve this modified problem.

9. Modify MEMOIZED-CUT-ROD to return not only the value but the actual solution, too.

10. The Fibonacci numbers are defined by the following recurrence:

$$F_n = \begin{cases} 0, & \text{for } n = 0; \\ 1, & \text{for } n = 1; \\ F_{n-1} + F_{n-2}, & \text{for } n \geq 2. \end{cases}$$

    Give an $O(n)$-time dynamic-programming algorithm to compute the n'th Fibonacci number. Draw the subproblem graph. How many vertices and edges are in the graph?