CS F364: Design & Analysis of Algorithms

Lecture-kt19: Linear Programming

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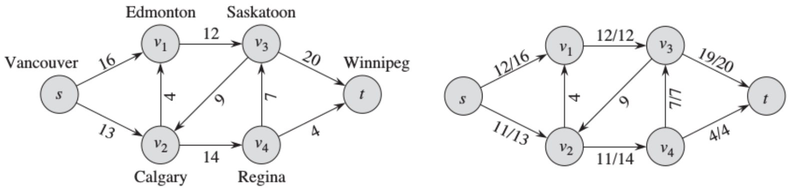
Recap: Maximum Flow

Edge represents capacity in terms of material that can flow through it



Problem is to

Find the Maximum Possible Flow in the network from *Vancouver* to *Winnipeg*; maintaining **flow conservation** ($\forall u \in V - \{s, t\}$, we require $\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$)

- Ford-Fulkerson method
- Edmonds-Karp algorithm (takes $O(|V| \times |E|^2)$ time)

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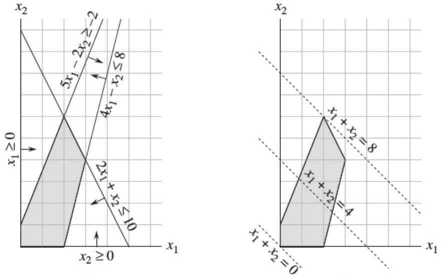

(a) (b)

Standard form

In standard form, we are given n real numbers $c_1, c_2, ..., c_n$ and m real numbers $b_1, b_2, ..., b_m$ along with $m \times n$ real numbers a_{ij} for $i \in \{1, 2, .., m\}$ and $j \in \{1, 2, .., n\}$

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This is equivalent to say

- Maximize $c^T x$
- Subject to
  1. $Ax \leq b$
  2. $x \geq 0$

Standard form example

Convert following to standard form

- Minimize $-2x_1 + 3x_2 + x_3$
- Subject to
  1. $x_1 + x_2 = 7$
  2. $x_1 - 2x_2 \leq 4$
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- basic variables and nonbasic variables

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & x_1, x_2, x_3 & & & \geq & 0 .
\end{array}$$

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& & & x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 & & & &
\end{array}$$

Slack form

- Thus we can define a slack form by a tuple (N, B, A, b, c, v) such that
- $z = v + \sum_{j \in N} c_j x_j$
- $x_i = b_i - \sum_{j \in N} a_{ij} x_j$ for $i \in B$

For example, in the slack form

$$\begin{aligned} z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\ x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\ x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\ x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} , \end{aligned}$$

we have $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$,

$$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix},$$

$$c = \begin{pmatrix} c_3 & c_5 & c_6 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -1/6 & -1/6 & -2/3 \end{pmatrix}^{\mathrm{T}}, \text{ and } v = 28.$$

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- In particular, we raise it until some basic variable becomes 0. We then rewrite the slack form, exchanging the roles of that basic variable and the chosen nonbasic variable.
- Its running time is not polynomial in the worst case

Simplex algorithm example

$$\begin{array}{lrcrcrcl}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{ect to} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
& 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
& & x_1, x_2, x_3 & & & & \geq & 0 .
\end{array}$$

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z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
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x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 .
\end{array}$$

Thank You!

Thank you very much for your attention! (Reference[1])

Queries ?

[1] [1] Book - *Introduction to Algorithm*, By THOMAS H. CORMEN, CHARLES E. LEISERSON, RONALD L. RIVEST, CLIFFORD STEIN