CS F364: Design & Analysis of Algorithms

Lecture-03: Asymptotic Notation + Master Method

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Recap: Correctness of Algorithm

Correct initialization, maintaining invariance in each iteration and terminate

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Insertion Sort

- Items A[1 .. j] are sorted

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Merge sort

- Sub array A[p..k-1] contains k-p smallest elements of L[1..n_1+1] and R[1..n_2+1] in sorted order.
- Also L[i] and R[j] are the smallest elements of their array.

Asymptotic Notation Θ

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$\Theta(g(n)) = \{f(n)$: there exists positive constants c_1, c_2 and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0\}$

- $\Theta(x) =$

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- $\Theta(x) = \{3x,$

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- $\Theta(x) = \{3x, 5x + 4, ...\}$
- $\Theta(\log x) =$

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- $\Theta(x) = \{3x, 5x + 4, ...\}$
- $\Theta(\log x) = \{4 \log x, 5 \log(x^3),$

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- $\Theta(x) = \{3x, 5x + 4, ...\}$
- $\Theta(\log x) = \{4 \log x, 5 \log(x^3), 5 \log(x^3) + 2, ...\}$
- We write $5x + 4 = \Theta(x)$ to mean $5x + 4 \in \Theta(x)$

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$$\lim_{n \to \infty} f(n)/g(n) = 0$$

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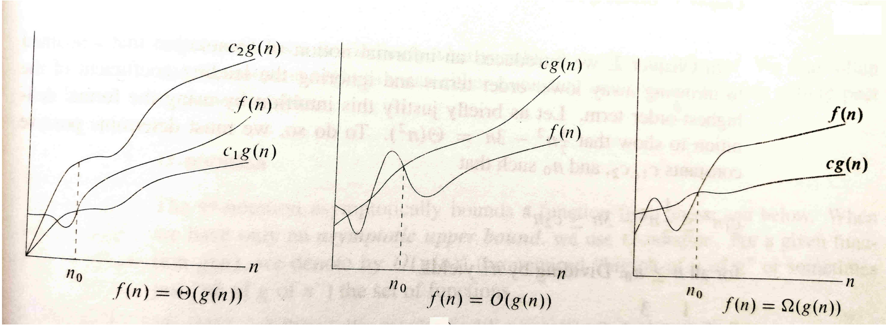
$$\lim_{n\to\infty} f(n)/g(n) = \infty$$

Intuition



$c_2g(n)$
$f(n)$
$c_1g(n)$
n_0
n
$f(n) = \Theta(g(n))$
$cg(n)$
$f(n)$
n_0
n
$f(n) = O(g(n))$
$f(n)$
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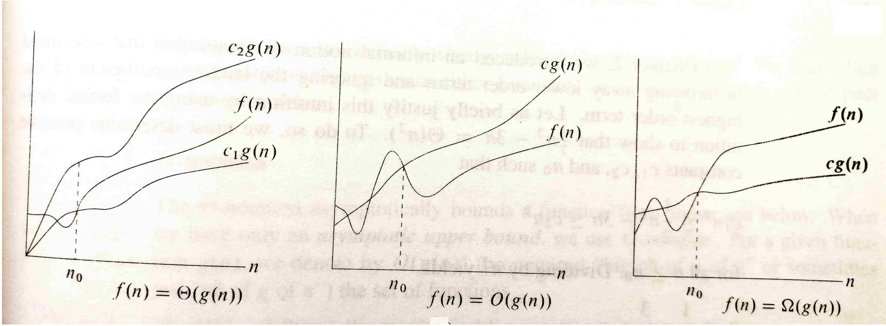
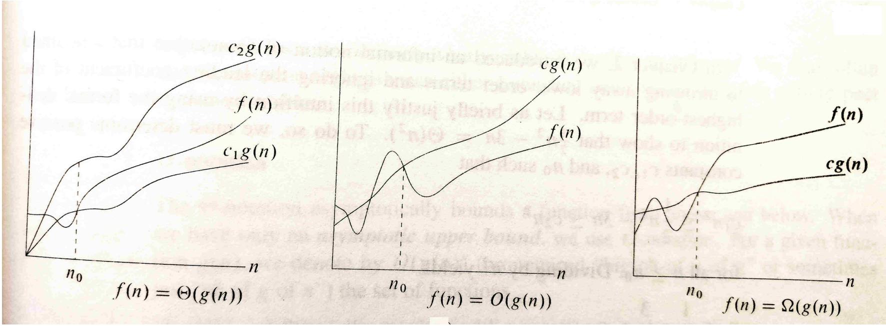
n_0
n
$f(n) = \Omega(g(n))$

Intuition



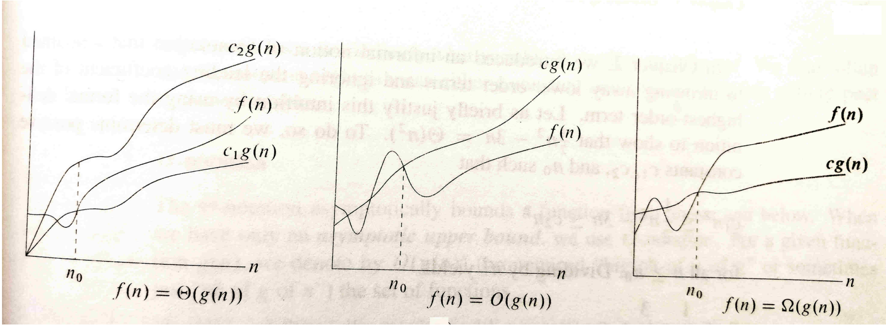
- Show max$(f(n), g(n))$ = $\Theta(f(n) + g(n))$

Intuition



- Show $\max(f(n), g(n)) = \Theta(f(n) + g(n))$
- Show $(n + a)^b = \Theta(n^b)$

Intuition



- Show $\max(f(n), g(n)) = \Theta(f(n) + g(n))$
- Show $(n + a)^b = \Theta(n^b)$
- Is $2^{n+1} = O(2^n)$

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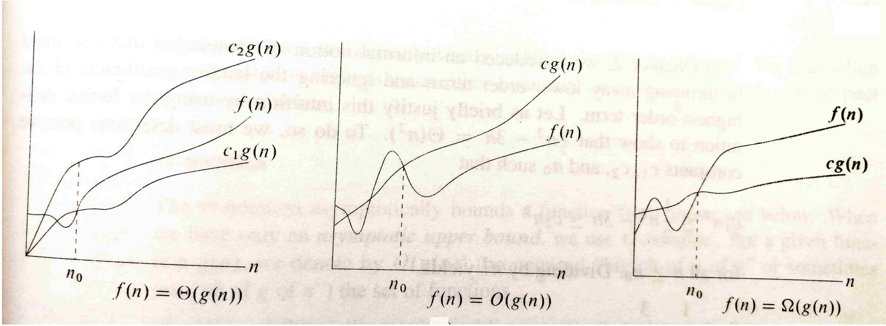

- Show $\max(f(n), g(n)) = \Theta(f(n) + g(n))$
- Show $(n + a)^b = \Theta(n^b)$
- Is $2^{n+1} = O(2^n)$
- $o(g(n)) \cap \omega(g(n))$ is an empty set
- Rank following functions based on their order of growth
  $\log\log n$, $2^{\log n}$, $\sqrt{\log n}$, e^n, $n^{\log n}$, $3n^3 + 5$

Recurrence relation

Equations of the form

$$T(n) = \begin{cases} \Theta(1) & \text{if } x \leq c \\ aT(n/b) + f(n) & \text{otherwise} \end{cases}$$

How to solve?

1. Substitution: guess the solution and test
2. Iteration: convert into summation and apply bounds
3. Master method

Substitution

Consider equation

$T(n) = 2T(\lfloor n/2 \rfloor) + n$

Let we guess the solution to be $T(n) = O(n \log n)$

$$
\begin{aligned}
T(n) &\leq 2(c\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor)) + n && (1)\\
&\leq cn \log(n/2) + n && (2)\\
&= cn \log(n) - cn \log 2 + n && (3)\\
&= cn \log(n) - cn + n && (4)\\
&\leq cn \log(n) && (5)
\end{aligned}
$$

As long as $c > 1$

Iteration

Consider equation

$T(n) = 3T(\lfloor n/4 \rfloor) + n$

$$
\begin{aligned}
T(n) &= n + 3T(\lfloor n/4 \rfloor) && (6)\\
&= n + 3(\lfloor n/4 \rfloor + 3T(\lfloor n/16 \rfloor)) && (7)\\
&= n + 3(\lfloor n/4 \rfloor + 3(\lfloor n/16 \rfloor + 3T(\lfloor n/64 \rfloor)) && (8)\\
&= n \sum_{i=0}^{\infty} (3/4)^i + \Theta(3^{\log_4 n}) && (9)\\
&= 4n + o(n) && (10)\\
&= O(n) && (11)\\
& && (12)
\end{aligned}
$$

Iteration stops when $\lfloor n/4^i \rfloor = 1$ that is $i = \log_4 n$

Master method

When $T(n) = aT(n/b) + f(n) \qquad a \geq 1, b > 1$

Let $\epsilon > 0$ be a constant

- If $f(n) = O(n^{log_b a-\epsilon})$ then $T(n) = \Theta(n^{log_b a})$
- If $f(n) = \Theta(n^{log_b a})$ then $T(n) = \Theta(n^{log_b a} \log n)$
- If $f(n) = \Omega(n^{log_b a+\epsilon})$ then $T(n) = \Theta(f(n))$
  provided if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.

Master method

When $T(n) = aT(n/b) + f(n) \qquad a \geq 1, b > 1$

Let $\epsilon > 0$ be a constant

- If $f(n) = O(n^{log_b a - \epsilon})$ then $T(n) = \Theta(n^{log_b a})$
- If $f(n) = \Theta(n^{log_b a})$ then $T(n) = \Theta(n^{log_b a} \log n)$
- If $f(n) = \Omega(n^{log_b a + \epsilon})$ then $T(n) = \Theta(f(n))$
  provided if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.

Solve following recurrences

- $T(n) = 9T(n/3) + n$
- $T(n) = T(2n/3) + 1$
- $T(n) = 3T(n/4) + n \log n$
- $T(n) = 2T(n/2) + n \log n$

Thank You!

Thank you very much for your attention!

Queries ?