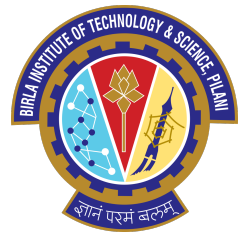
# CS F364: Design & Analysis of Algorithms

## Lecture-kt04: Recurrence Relation + Quick sort




**Dr. Kamlesh Tiwari**,
Assistant Professor,
Department of Computer Science and Information Systems,
BITS Pilani, Rajasthan-333031 INDIA

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# Recap: Asymptotic Notation

| | | |
|---|---|---|
| $\Theta(g(n))$ | $=$ | $\{f(n)$ : there exists positive constants $c_1, c_2$ and $n_0$ such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0\}$ |
| $O(g(n))$ | $=$ | $\{f(n)$ : there exists positive constants $c$ and $n_0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$ |
| $o(g(n))$ | $=$ | $\{f(n)$ : for any positive constants $c$, there exists a constant $n_0 > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0\}$ |
| $\Omega(g(n))$ | $=$ | $\{f(n)$ : there exists positive constants $c$ and $n_0$ such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0\}$ |
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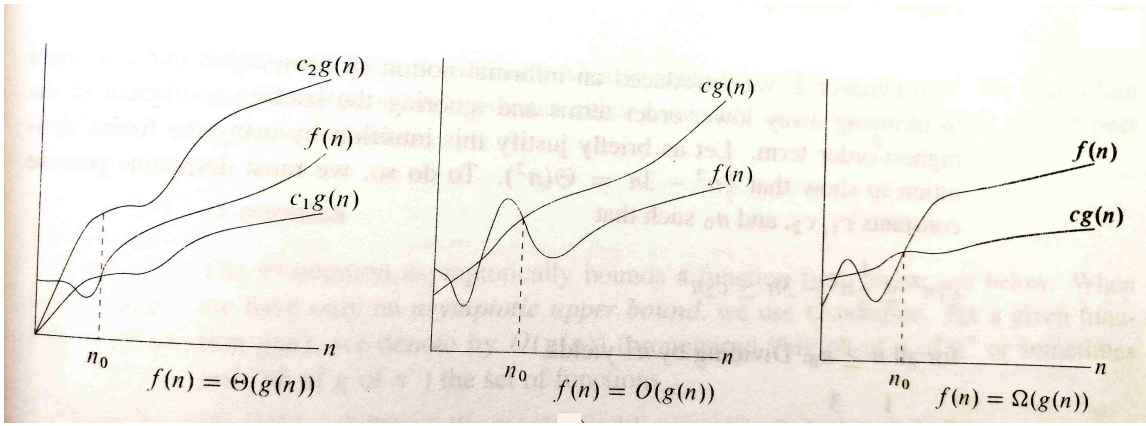

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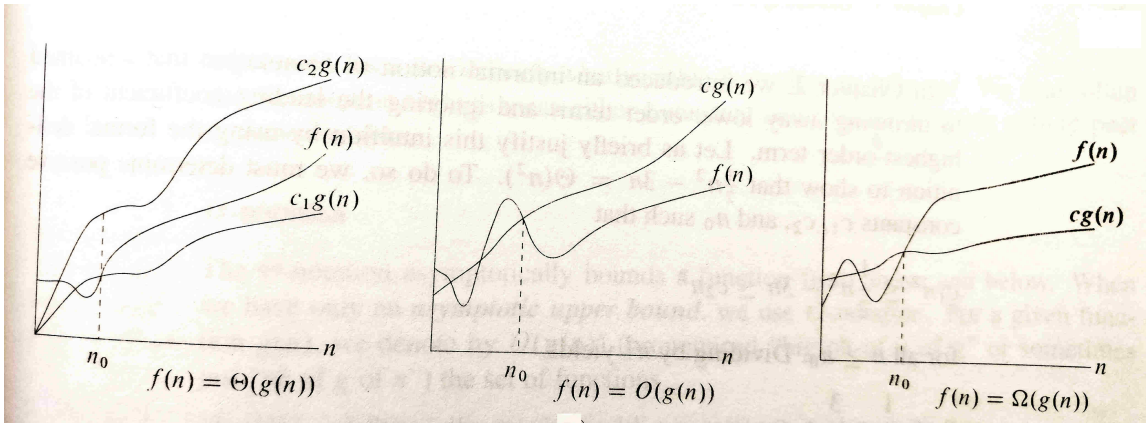

# Recurrence relation

## Equations like

$$T(n) = \begin{cases} \Theta(1) & \text{if } x \leq c \\ aT(n/b) + f(n) & \text{otherwise} \end{cases}$$

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Or $T(n) = a_1 T(n - k_1) + b_2 T(n - k_2) + ... + f(n)$

**How to solve?** Three approaches

1. Substitution: guess the solution and test
2. Iteration: convert into summation and apply bounds
3. Master method

# Substitution

## Consider equation

$T(n) = 2T(\lfloor n/2 \rfloor) + n$

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$$\begin{aligned} T(n) &\leq 2(c\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor)) + n \\ &\leq cn \log(n/2) + n \end{aligned} \tag{1}$$

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$$\begin{aligned} T(n) &\leq 2(c\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor)) + n && (1) \\ &\leq cn \log(n/2) + n && (2) \\ &= cn \log(n) - cn \log 2 + n \end{aligned}$$

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&= cn \log(n) - cn + n
\end{aligned}
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&= cn \log(n) - cn + n && (4)\\
&\leq cn \log(n) && (5)
\end{aligned}
$$

As long as $c > 1$ and for all $n > 1$

# Iteration

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$$T(n) = n + 3T(\lfloor n/4 \rfloor) \tag{6}$$

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$$
\begin{aligned}
T(n) &= n + 3T(\lfloor n/4 \rfloor) && (6)\\
&= n + 3(\lfloor n/4 \rfloor + 3T(\lfloor n/16 \rfloor)) && (7)
\end{aligned}
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&= n + 3(\lfloor n/4 \rfloor + 3(\lfloor n/16 \rfloor + 3T(\lfloor n/64 \rfloor))) && (8)
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&= n\sum_{i=0}^{\infty}(3/4)^i + \Theta(3^{\log_4 n}) && (9)
\end{aligned}
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&= n\sum_{i=0}^{\infty}(3/4)^i + \Theta(3^{\log_4 n}) && (9)\\
&= 4n + o(n) && (10)\\
&= O(n) && (11)\\
& && (12)
\end{aligned}
$$

Iteration stops when $\lfloor n/4^i \rfloor = 1$ that is $i = \log_4 n$

# Master method

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- If $f(n) = \Omega(n^{log_b a + \epsilon})$ then $T(n) = \Theta(f(n))$
  provided if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large $n$.

  **Note: Regularity condition must always be checked in case-3.**

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**Solve following recurrences**

- $T(n) = 9T(n/3) + n$

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# Quick Sort

- John Von Neumann proposed *merge sort* in 1945 which takes $n \log_2 n$ time to sort $n$ items in average and worst case.
- In 1960 C.A.R. Hoare proposed *quick sort* that takes $1.39 \times n \log_2 n$ time on average and $n(n-1)$ time in worst case.
- Quick sort is more popular because it practically always behaves like average case as the number of items $n$ increases.
- 1000 time execution of randomized quick sort on randomly selected items

| | Number of items $n =$ | | | | |
|---|---|---|---|---|---|
| Number of times the runtime exceed average behavior | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
| 10% | 190 | 49 | 22 | 10 | 3 |
| 20% | 28 | 17 | 12 | 3 | 0 |
| 50% | 2 | 1 | 1 | 0 | 0 |
| 100% | 0 | 0 | 0 | 0 | 0 |

# Thank You!

**Thank you very much for your attention!**

**Queries ?**