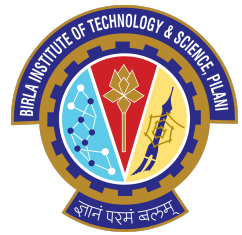
CS F364: Design & Analysis of Algorithms

Lecture-kt05: Randomized Quick Sort



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Recap: Master method

When $T(n) = aT(n/b) + f(n)$ $\quad a \geq 1, b > 1$

Let $\epsilon > 0$ be a constant

1. If $f(n) = O(n^{log_b a - \epsilon})$ then $T(n) = \Theta(n^{log_b a})$
2. If $f(n) = \Theta(n^{log_b a})$ then $T(n) = \Theta(n^{log_b a} \log n)$
3. If $f(n) = \Omega(n^{log_b a + \epsilon})$ then $T(n) = \Theta(f(n))$
   provided if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n. **Regularity condition must be checked in case-3.**

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Recap: Quick Sort

Which algorithm is better ?

	Best Case	Worst Case	Average Case
Algo-01	$n \log n$	$n \log n$	$n \log n$
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- Why quick sort is popular? it always behaves like average case as the number of items to be sorted sort increases
- 1000 time execution of randomized quick sort on randomly selected items

	Number of items $n =$				
Number of times the runtime exceed average behavior	10^2	10^3	10^4	10^5	10^6
10%	190	49	22	10	3
20%	28	17	12	3	0
50%	2	1	1	0	0
100%	0	0	0	0	0

Execution of Quick Sort

```
QUICKSORT(A, p, r)
1  if p < r
2        q = PARTITION(A, p, r)
3        QUICKSORT(A, p, q − 1)
4        QUICKSORT(A, q + 1, r)
```

```
PARTITION(A, p, r)
1  x = A[r]
2  i = p − 1
3  for j = p to r − 1
4       if A[j] ≤ x
5            i = i + 1
6            exchange A[i] with A[j]
7  exchange A[i + 1] with A[r]
8  return i + 1
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Randomized Quick Sort

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RANDOMIZED-QUICKSORT(A, p, r)
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```
RANDOMIZED-PARTITION(A, p, r)
1  i = RANDOM(p, r)
2  exchange A[r] with A[i]
3  return PARTITION(A, p, r)
```

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- X_{ij} can either take a value 0 or 1
- Expected number of comparison is

$$E[\sum_{i=1}^{n}\sum_{j>i} X_{ij}] = \sum_{i=1}^{n}\sum_{j>i} E[X_{ij}]$$

Randomized Quick Sort

- Let p_{ij} denote the probability of comparison between S_i and S_j. Then,

$$E[X_{ij}] = p_{ij} \times 1 + (1 - p_{ij}) \times 0 = p_{ij}$$

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&= O(n \ln n)
\end{aligned}
$$

as $H_n \sim \ln n + \Theta(1)$

Thank You!

Thank you very much for your attention!

Queries ?