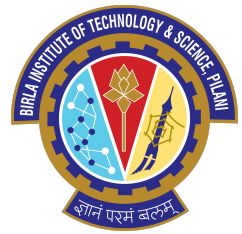
CS F364: Design & Analysis of Algorithms

Lecture-kt06: Sorting, Lower Bounds and Linear Time Solution



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Recap: Analysis of Randomized Quick Sort

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RANDOMIZED-QUICKSORT(A, p, r)
1  if p < r
2      q = RANDOMIZED-PARTITION(A, p, r)
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PARTITION(A, p, r)
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Let p_{ij} be probability of comparison between $S_i \leftrightarrow S_j$

$$E[X_{ij}] = p_{ij} \times 1 + (1 - p_{ij}) \times 0 = p_{ij}$$

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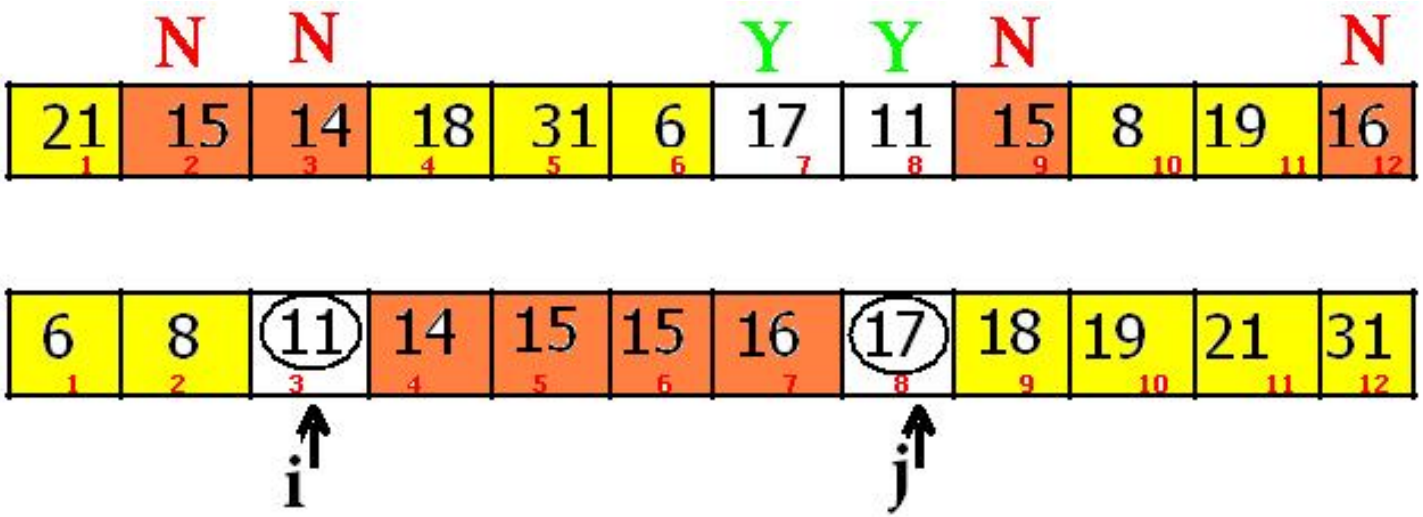

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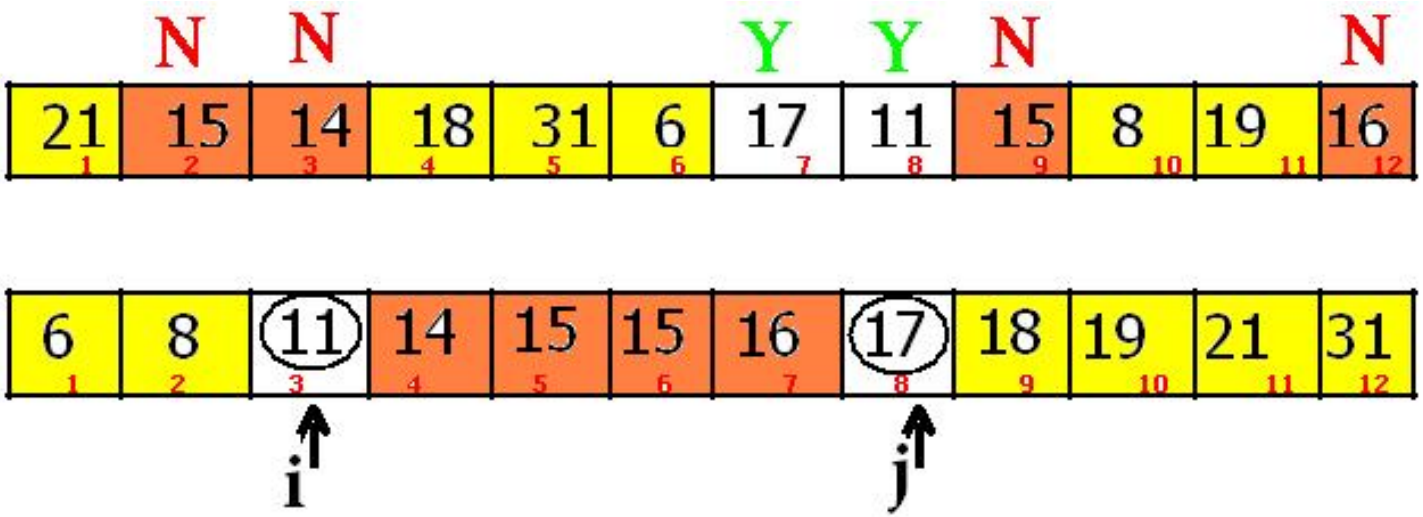

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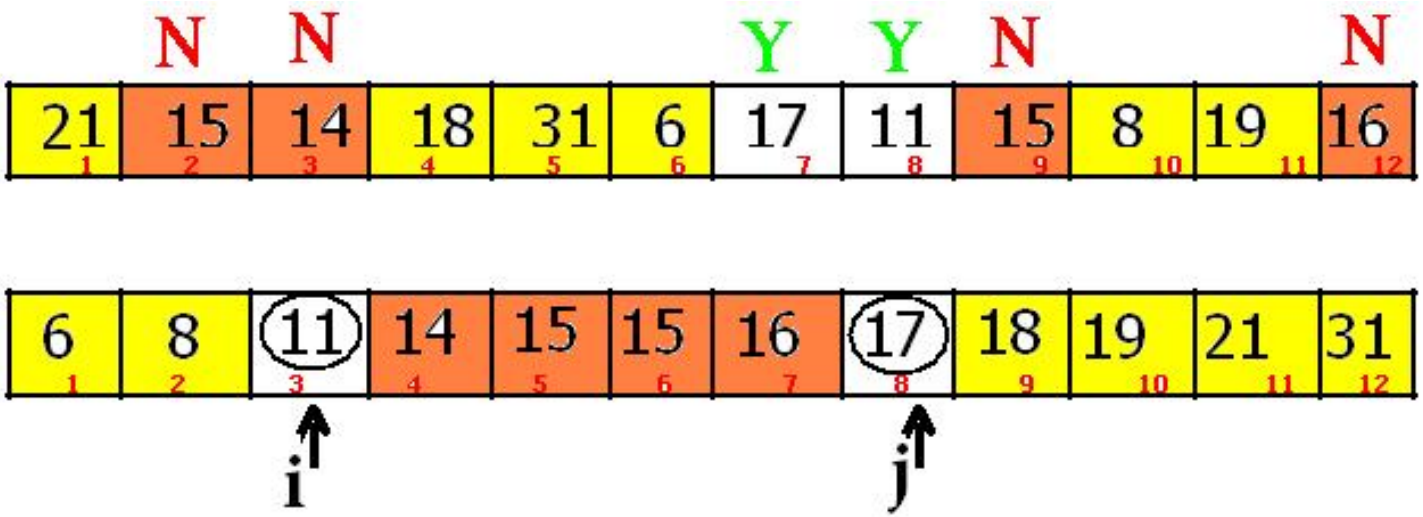

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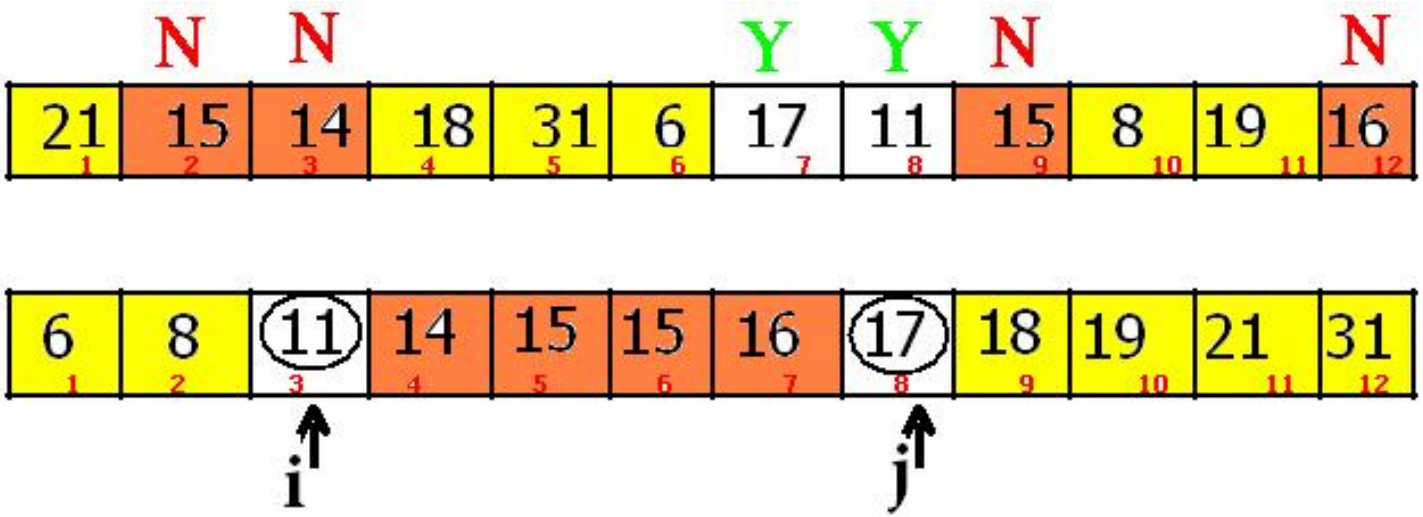

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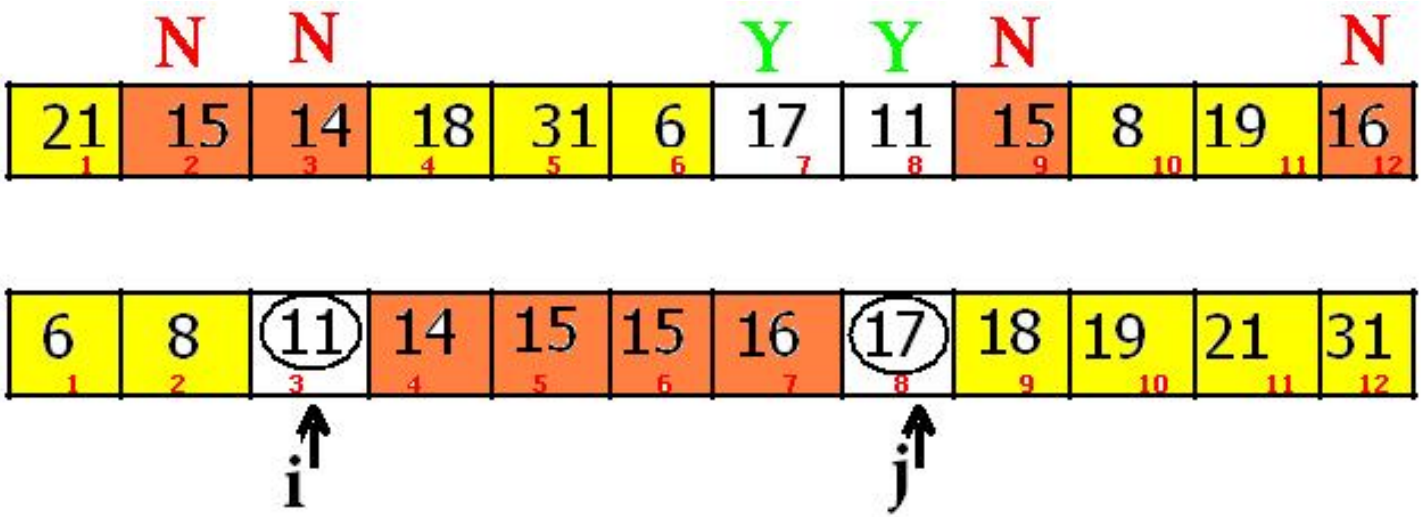

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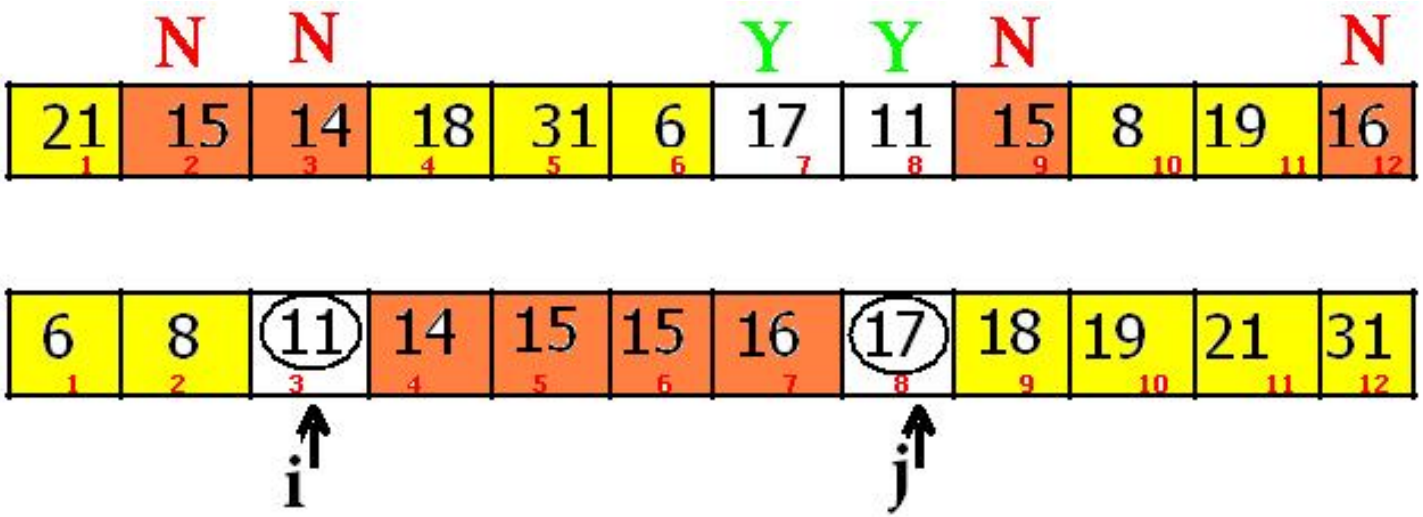

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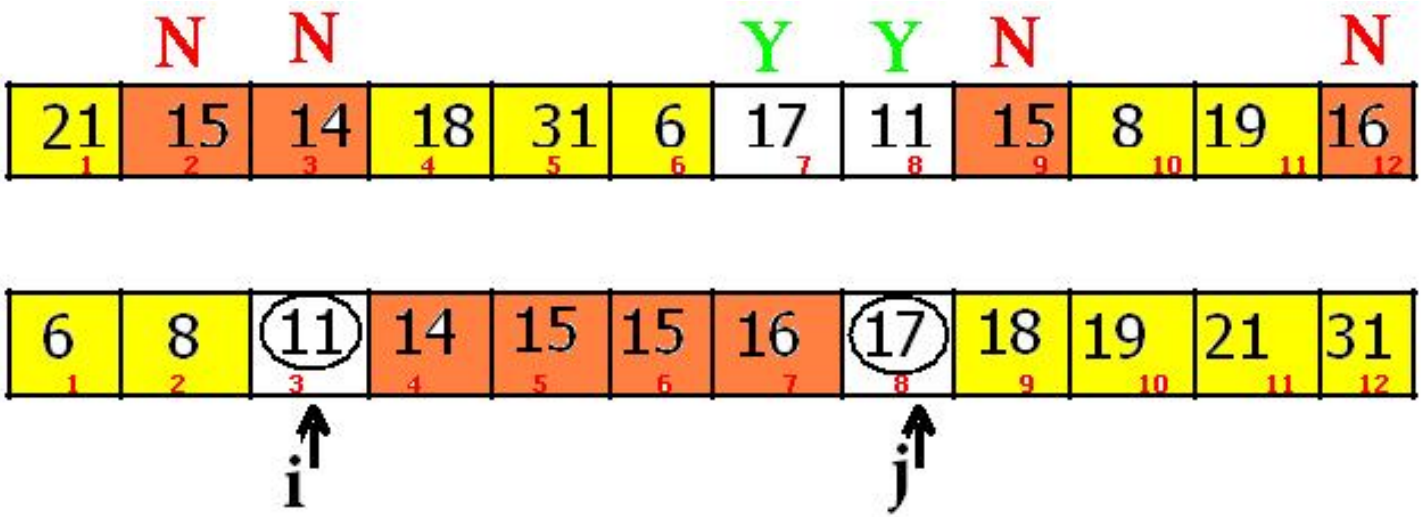

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$$\leq 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k} = 2nH_n = O(n \ln n)$$

Decision Tree Model

Sort three items a_1, a_2, a_3

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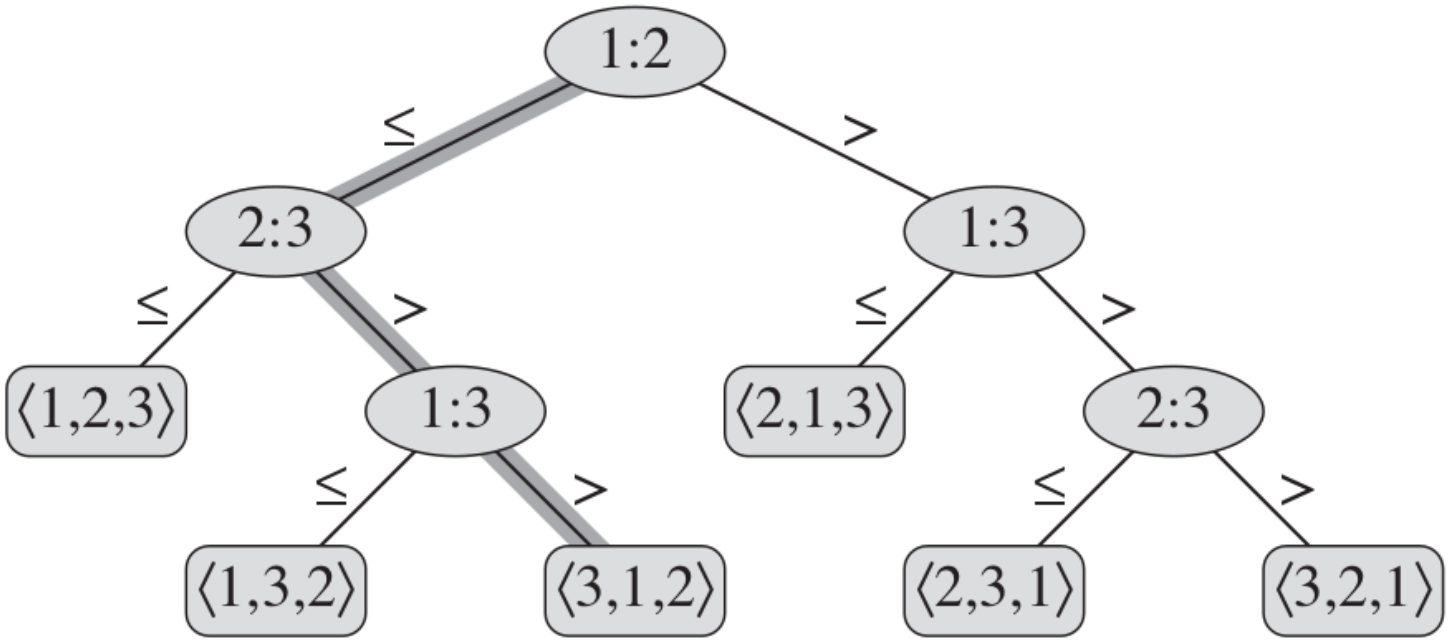

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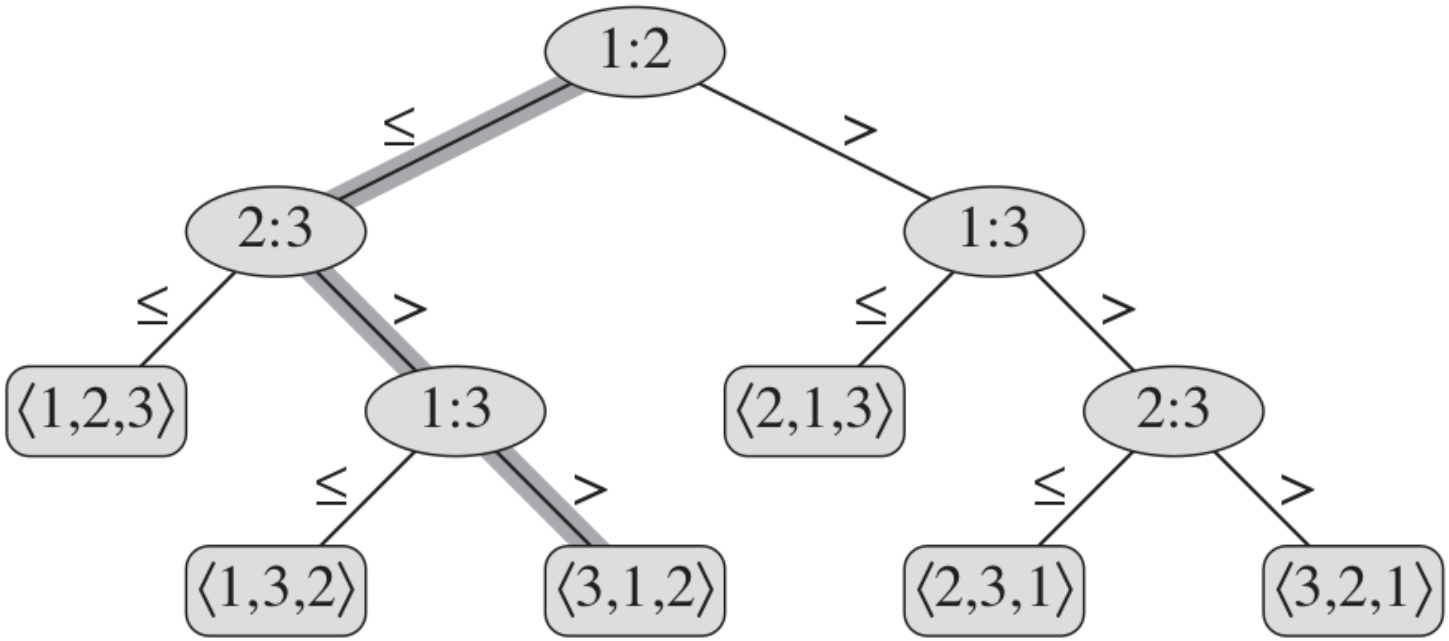

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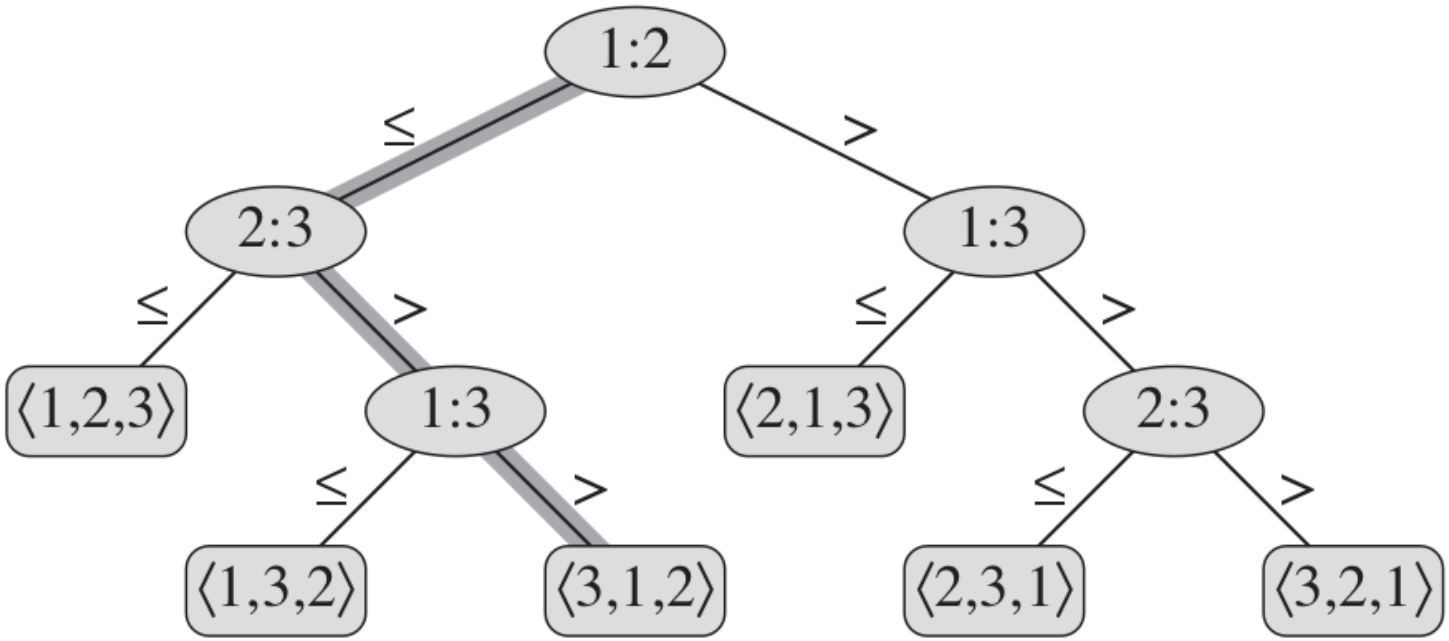

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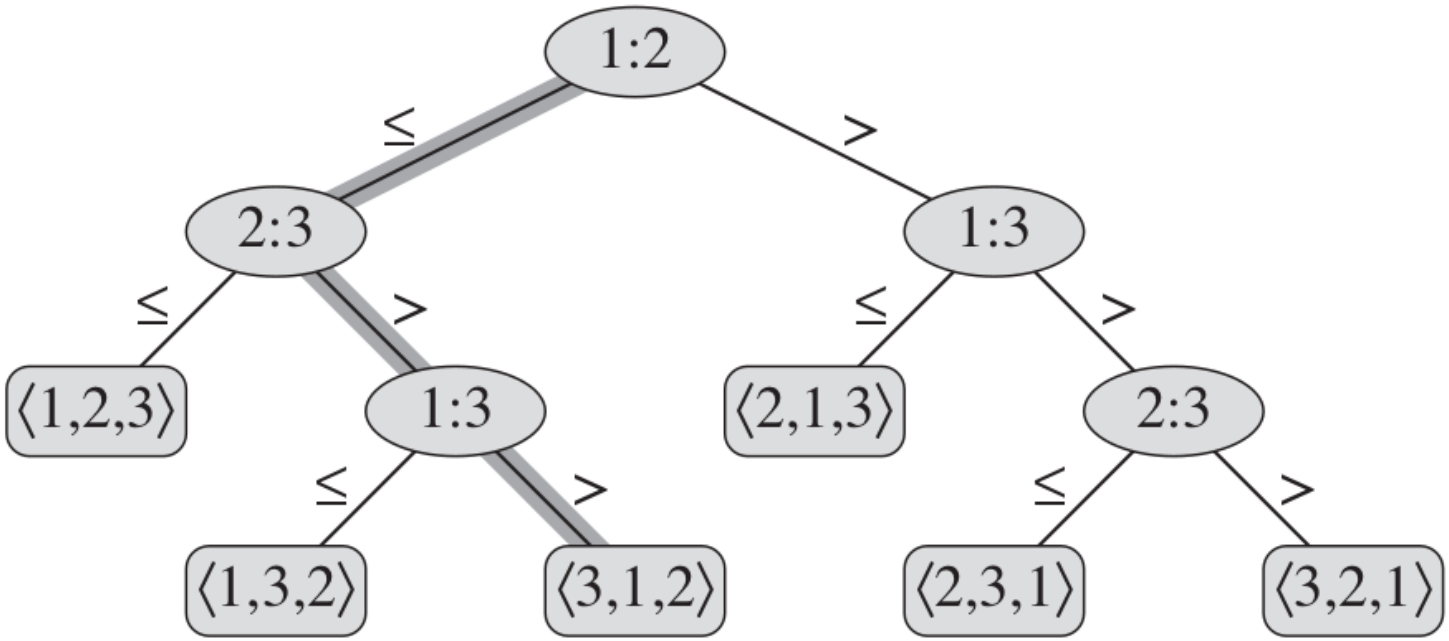

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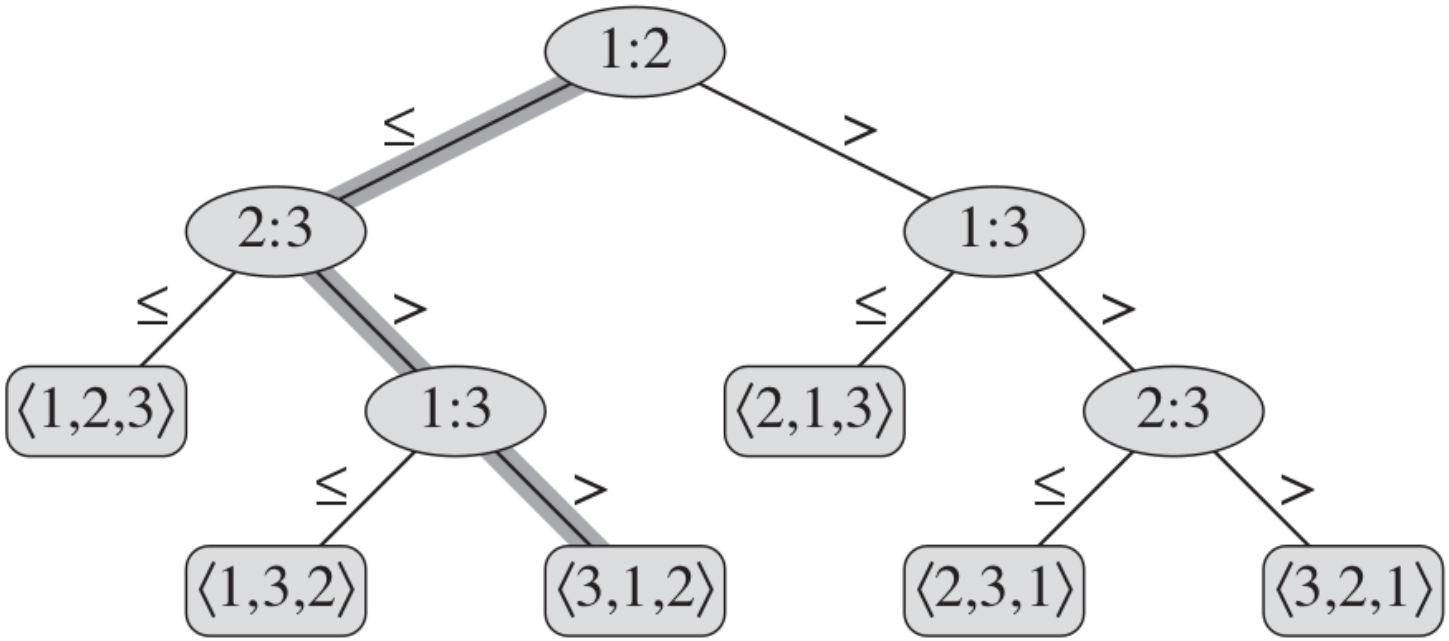

Is it always to be a binary tree?
What is worst case time taken by this algorithm? $O(height)$
How many leaves would be there with 4 items?

Lower Bound

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$$h = \Omega(n\log(n))$$

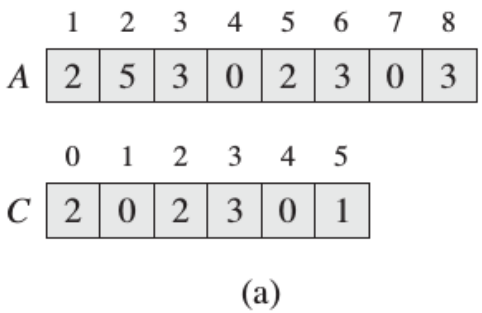
Still Wish $O(n)$? use Counting Sort

COUNTING-SORT(A, B, k)

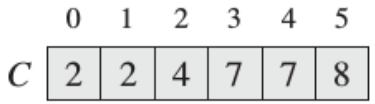
```
1   let C[0..k] be a new array
2   for i = 0 to k
3       C[i] = 0
4   for j = 1 to A.length
5       C[A[j]] = C[A[j]] + 1
6   // C[i] now contains the number of elements equal to i.
7   for i = 1 to k
8       C[i] = C[i] + C[i − 1]
9   // C[i] now contains the number of elements less than or equal to i.
10  for j = A.length downto 1
11      B[C[A[j]]] = A[j]
12      C[A[j]] = C[A[j]] − 1
```

Sort: 2, 5, 3, 0, 2, 3, 0, 3

Counting Sort in action



(a)



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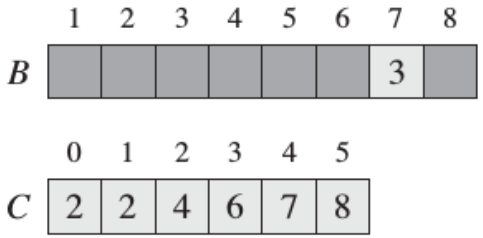

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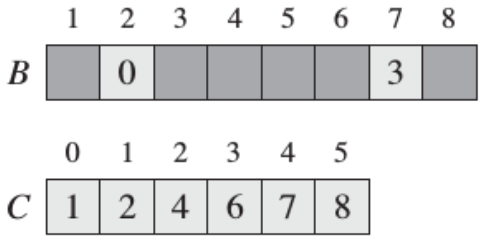

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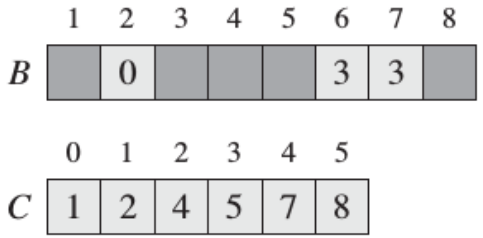

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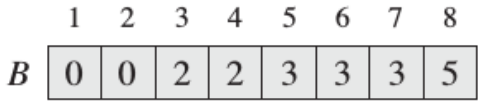
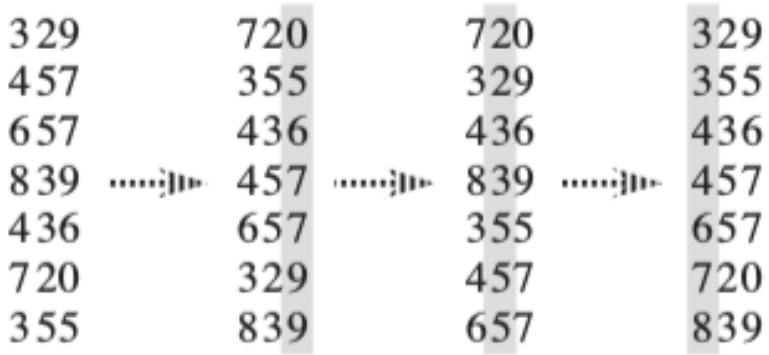


(f)

Radix Sort

Use a stable sorting algorithm to sort array A on digit (1 to d)



Bucket Sort

BUCKET-SORT(A)

```
1  let B[0..n − 1] be a new array
2  n = A.length
3  for i = 0 to n − 1
4       make B[i] an empty list
5  for i = 1 to n
6       insert A[i] into list B[⌊nA[i]⌋]
7  for i = 0 to n − 1
8       sort list B[i] with insertion sort
9  concatenate the lists B[0], B[1], . . . , B[n − 1] together in order
```

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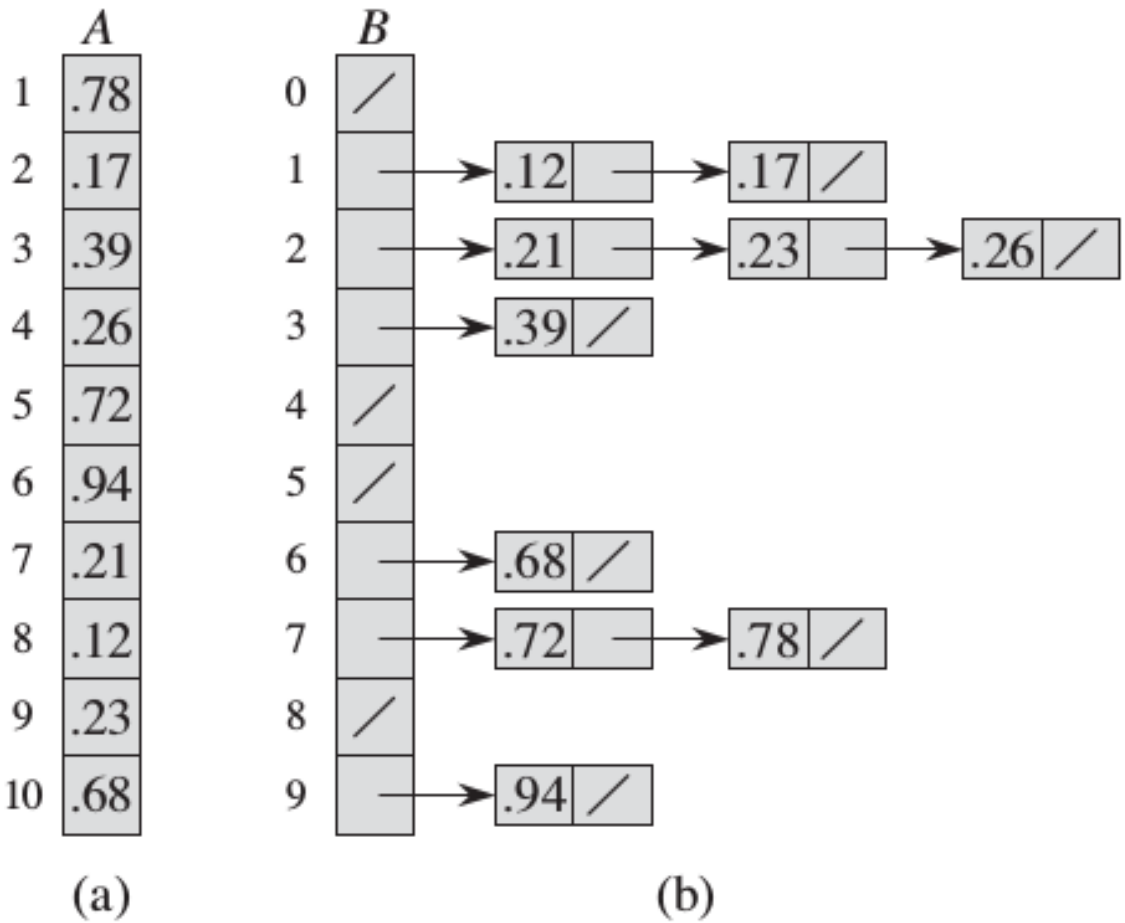
A
1 .78
2 .17
3 .39
4 .26
5 .72
6 .94
7 .21
8 .12
9 .23
10 .68
(a)
B
0
1 .12 .17
2 .21 .23 .26
3 .39
4
5
6 .68
7 .72 .78
8
9 .94
(b)

Thank You!

Thank you very much for your attention!

Queries ?