CS F364: Design & Analysis of Algorithms

Lecture-kt09: Hashing (contd.) + Binary Search Tree

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Assuming **load factor** $\alpha = n/m$ to be a constant under **simple uniform hashing** all the operations becomes $O(1)$ time

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- Let H be finite collection of hash function that map a given universe U of keys into the range $\{0, 1, 2, ..., m\}$. such a collection is said to be **universal** if for each pair of distinct keys $x, y \in U$, the number of hash functions $h \in H$ for which $h(x) = h(y)$ is precisely $|H|/m$.

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Therefore, H is universal.

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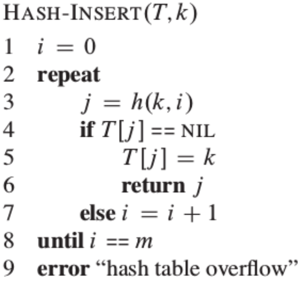

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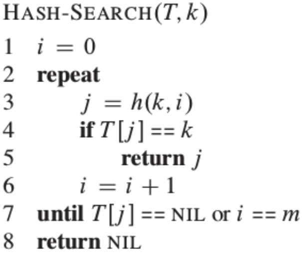

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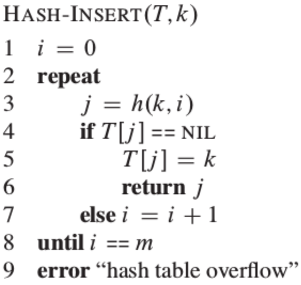

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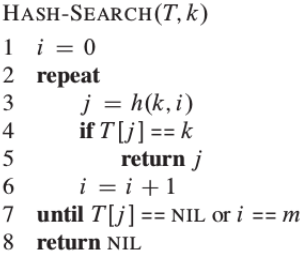

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Deletion from an open-address hash table is difficult.

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- ***Double Hashing:***
  $h(k,i) = (h_1(k) + i \times h_2(k)) \bmod m$

In open addressing an unsuccessful search has probe $1/(1-\alpha)$ and in successful one is $\frac{1}{\alpha} \ln \frac{1}{1-\alpha} + \frac{1}{\alpha}$. Where $\alpha = n/m < 1$.

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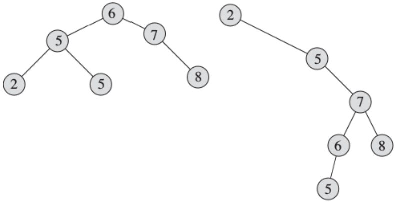

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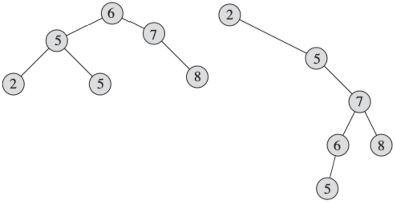

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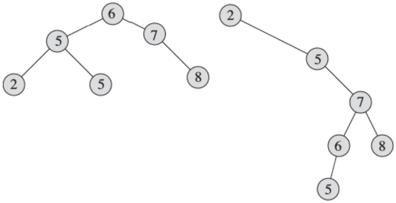

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- In order walk is monotonous

Thank You!

Thank you very much for your attention!

Queries ?